

Large deviations in 1D systems: mapping non-equilibrium to equilibrium

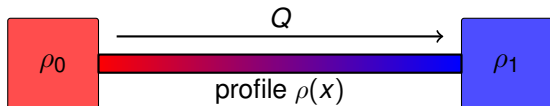
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Motivations



$$\text{Prob}(\mathcal{C}) \propto \exp \left\{ - \frac{\text{energy}(\mathcal{C})}{\text{temperature}} \right\} \quad \text{cannot describe}$$

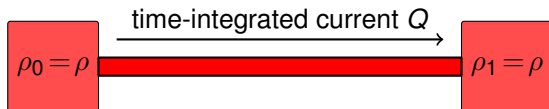
- Non-equilibrium steady-state

$$\text{Prob}[\rho(x)]$$

- Equilibrium fluctuations of dynamical observables

$$\text{Prob}[Q]$$

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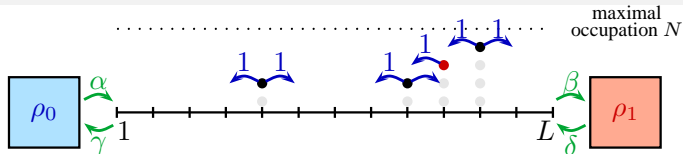
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Exclusion processes



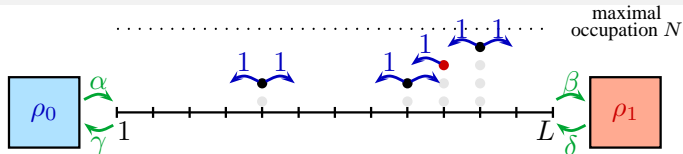
- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution

$$\partial_t P(\{n_i\}) = \sum_{n'_i} [W(n'_i \rightarrow n_i)P(\{n'_i\}) - W(n_i \rightarrow n'_i)P(\{n_i\})]$$

- Large deviation function of the time-integrated current Q

$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \quad (\Leftrightarrow \text{determining } P(Q))$$

Exclusion processes



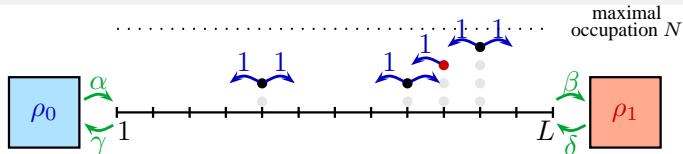
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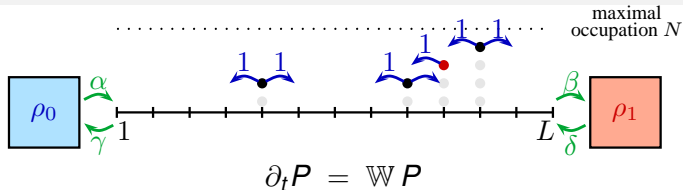
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Operator representation

[Schütz & Sandow PRE 49 2726]



$$\mathbb{W} = \sum_{1 \leq k \leq L-1} [S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k (1 - \hat{n}_{k+1}) - \hat{n}_{k+1} (1 - \hat{n}_k)]$$

$$+ \alpha [S_1^+ - (1 - \hat{n}_1)] + \gamma [S_1^- - \hat{n}_1]$$

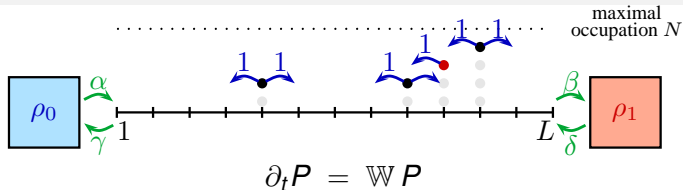
$$+ \delta [S_L^+ - (1 - \hat{n}_L)] + \beta [S_L^- - \hat{n}_L]$$

S^\pm and **creation and annihilation operators:**

$$S^+ |n\rangle = (N - n) |n + 1\rangle \quad S^- |n\rangle = n |n - 1\rangle$$

Operator representation

[Schütz & Sandow PRE 49 2726]



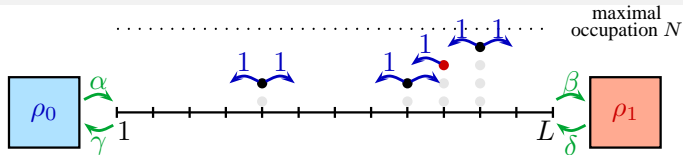
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S^\pm and **creation and annihilation operators:**

$$S^+ |n\rangle = (N - n) |n + 1\rangle \quad S^- |n\rangle = n |n - 1\rangle$$

S^\pm and $S^z = \hat{n} - \frac{N}{2}$ are **spin operators** (with $j = \frac{N}{2}$)

Large deviation function



$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \quad \text{with} \quad \psi(s) = \max \text{Sp } \mathbb{W}(s)$$

$$\begin{aligned} \mathbb{W}(s) = & \sum_{1 \leq k \leq L-1} [S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k(1 - \hat{n}_{k+1}) - \hat{n}_{k+1}(1 - \hat{n}_k)] \\ & + \alpha [S_1^+ - (1 - \hat{n}_1)] + \gamma [S_1^- - \hat{n}_1] \\ & + \delta [S_L^+ e^s - (1 - \hat{n}_L)] + \beta [S_L^- e^{-s} - \hat{n}_L] \end{aligned}$$

1st step: mapping

Mapping non-eq to eq

[Imparato, VL, van Wijland, *PTPS* **184** 276]

Large deviations of the current

$$\psi(\mathbf{s}) = \max_{\text{Sp}} \mathbb{W}(\mathbf{s})$$

$$\mathbb{W}(\mathbf{s}) = \overbrace{\sum_{1 \leq k \leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}} + \alpha [S_1^+ - (1 - \hat{n}_1)] + \gamma [S_1^- - \hat{n}_1] + \delta [S_L^+ e^{\mathbf{s}} - (1 - \hat{n}_L)] + \beta [S_L^- e^{-\mathbf{s}} - \hat{n}_L]$$

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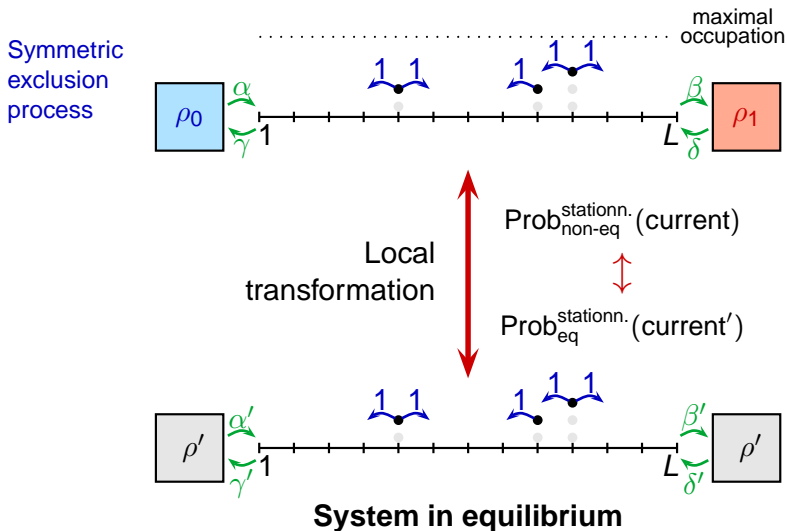
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Local transformation

$$\mathcal{Q}^{-1} \mathbb{W}(\mathbf{s}) \mathcal{Q} = \sum_{1 \leq k \leq L-1} \vec{S}_k \cdot \vec{S}_{k+1} + \alpha' [S_1^+ - (1 - \hat{n}_1)] + \gamma' [S_1^- - \hat{n}_1] + \delta' [S_L^+ e^{\mathbf{s}'} - (1 - \hat{n}_L)] + \beta' [S_L^- e^{-\mathbf{s}'} - \hat{n}_L]$$

describes contact with reservoirs of same densities

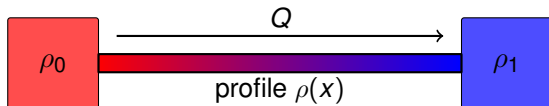
For the current

[Imparato, VL, van Wijland, **PRE** 80 011131]

2nd step: macroscopic limit

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

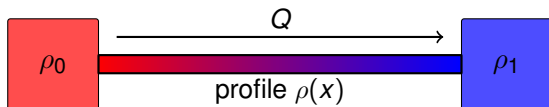


A reminder: propagator in quantum mechanics

$$\langle \text{final} | e^{it\mathbb{H}} | \text{initial} \rangle$$

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

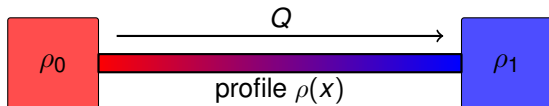


A reminder: propagator in quantum mechanics

$$\langle \text{final} | e^{it\mathbb{H}} | \text{initial} \rangle = \int dz_1 \dots dz_n \langle \text{final} | e^{i\Delta t\mathbb{H}} | \underline{z}_n \rangle \langle \underline{z}_{n-1} | e^{i\Delta t\mathbb{H}} | \underline{z}_{n-2} \rangle \dots \\ \dots \langle \underline{z}_1 | e^{i\Delta t\mathbb{H}} | \text{initial} \rangle$$

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

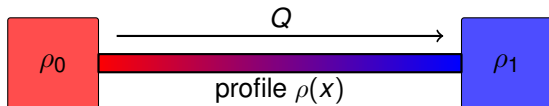


A reminder: propagator in quantum mechanics

$$\begin{aligned}
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 &\quad \dots \langle \underline{z}_1 | e^{i\Delta t\mathbb{H}} | \text{initial} \rangle \\
 &= \int \mathcal{D}p \mathcal{D}q \exp \left\{ i \frac{1}{\hbar} \underbrace{\mathcal{S}[p, q]}_{\text{action}} \right\}
 \end{aligned}$$

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

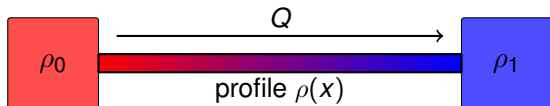


Using $SU(2)$ coherent states:

$$\langle \rho_f | e^{t\mathbb{W}} | \rho_i \rangle = \int_{\rho(0)=\rho_i}^{\rho(t)=\rho_f} \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp\{ \underbrace{L \mathcal{S}[\hat{\rho}, \rho]}_{\text{action}} \}$$

Macroscopic limit

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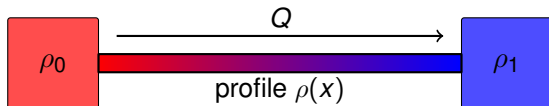
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$$\langle e^{-sQ} \rangle \sim \langle \rho_f | e^{t\mathbb{W}_s} | \rho_i \rangle = \int_{\rho(0)=\rho_i}^{\rho(t)=\rho_f} \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp\{L \underbrace{\mathcal{S}_s[\hat{\rho}, \rho]}_{\text{action}}\}$$

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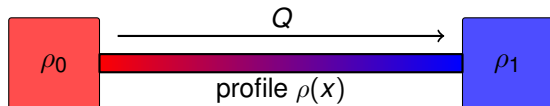
Same $\mathcal{S}_s[\hat{\rho}, \rho]$ as the MSR action of the Langevin evolution:

$$\partial_t \rho = -\partial_x [-\partial_x \rho + \xi]$$

$$\langle \xi(x, t) \xi(x', t') \rangle = \underbrace{\frac{1}{L} \rho(1 - \rho)}_{\text{density-dependent}} \delta(x' - x) \delta(t' - t)$$

Macroscopic limit

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One recovers the action of fluctuating hydrodynamics

[Spohn, Bertini De Sole Gabrielli Jona-Lasinio Landim]

We gain **microscopic boundary terms**

Macroscopic approach

[Imparato, VL, van Wijland, PRE **80** 011131]

$$\langle e^{-sQ} \rangle \sim \int \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp\{L \mathcal{S}_s[\hat{\rho}, \rho]\}$$

Fluctuations $\phi, \hat{\phi}$ around the saddle

$$\rho(\mathbf{x}, t) = \rho_c(\mathbf{x}) + \phi(\mathbf{x}, t)$$

$$\hat{\rho}(\mathbf{x}, t) = \hat{\rho}_c(\mathbf{x}) + \hat{\phi}(\mathbf{x}, t)$$

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Fluctuations $\phi, \hat{\phi}$ around the saddle

$$\rho(x, t) = \rho_c(x) + \phi(x, t)$$

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Mapping of non-eq. fluctuations $\phi, \hat{\phi}$ to eq. fluctuations $\phi', \hat{\phi}'$

$$\phi(x, t) = (\partial_x \hat{\rho}_c)^{-1} \phi'(x, t) + (\partial_x \rho_c)^{-1} \hat{\phi}'(x, t)$$

$$\hat{\phi}(x, t) = (\partial_x \hat{\rho}_c) \hat{\phi}'(x, t)$$

Result

[Imparato, VL, van Wijland, **PRE** 80 011131]

- Large deviation function

[Reminder: $\langle e^{-sQ} \rangle \sim e^{t\psi(s)}$]

$$\psi(s) = \underbrace{\frac{1}{L}\mu(s)}_{\text{saddle}} + \underbrace{\frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(s)\right)}_{\text{fluctuations}}$$

$$\mu(s) = -(\operatorname{arcsinh} \sqrt{\omega})^2 \quad \omega = (1 - e^s)(e^{-s}\rho_0 - \rho_1 - (e^{-s} - 1)\rho_0\rho_1)$$

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[Imparato, VL, van Wijland, **PRE** 80 011131]

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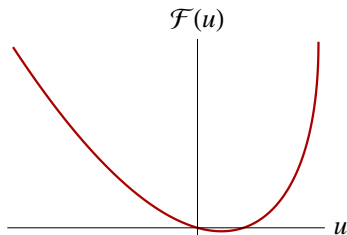
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$$\mathcal{F}(u) = \sum_{k \geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$



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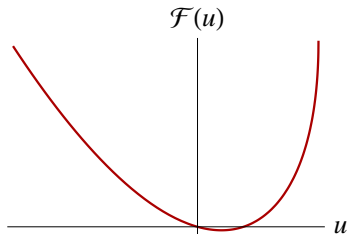
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- No transition:** $u < 0$
(neither for KMP)



Remark 1: [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
Driving field E

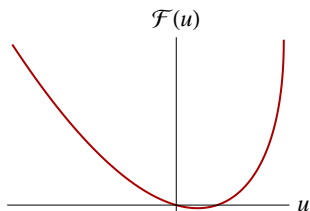
Remark 1: [Appert, Derrida, VL, van Wijland, PRE 78 021122]

Periodic boundary conditions
Driving field E

For the WASEP:
 $D = 1, \sigma = 2\rho(1 - \rho)$

Large deviation function

$$\psi(s) = \underbrace{\frac{1}{2}s(s - E)\frac{\langle Q^2 \rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\substack{\text{small fluctuations} \\ \text{(determinant)}}} \quad \text{with} \quad u = \underbrace{-s(s - E)\frac{\sigma\sigma''}{16D^2}}_{\text{can become } > 0}$$



Dynamical phase transition
between
stationary and non-stationary
profiles

Remark 2: Bethe Ansatz

[Appert, Derrida, VL et al. PRE **78** 021122]

SSEP: maximal occupation $N = 1$
Periodic boundary conditions

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SSEP: maximal occupation $N = 1$

Periodic boundary conditions

Bethe Ansatz:

- eigenvector of components

$$\sum_{\mathcal{P}} \mathcal{A}(\mathcal{P}) \prod_{i=1}^N [\zeta_{\mathcal{P}(i)}]^{x_i}$$

- eigenvalue

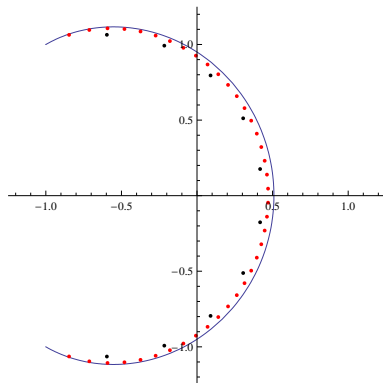
$$\psi(\mathbf{s}) = -2N + e^{-s} [\zeta_1 + \dots + \zeta_N] - e^s \left[\frac{1}{\zeta_1} + \dots + \frac{1}{\zeta_N} \right]$$

- Bethe equations

$$\zeta_i^L = \prod_{\substack{j=1 \\ j \neq i}}^N \left[- \frac{1 - 2e^{-s}\zeta_i + e^{-2s}\zeta_i\zeta_j}{1 - 2e^{-s}\zeta_j + e^{-2s}\zeta_i\zeta_j} \right]$$

Remark 2: Bethe Ansatz

[Appert, Derrida, VL et al. PRE **78** 021122]



Repartition of Bethe roots in the complex plane

Remark 2: Bethe Ansatz

[Appert, Derrida, VL et al. PRE **78** 021122]

- large deviation function

$$\psi(\mathbf{s}) = \underbrace{L\rho(1-\rho)\mathbf{s}^2}_{\text{continuum}} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{discreteness}} \quad \text{with} \quad u = -\frac{1}{2}L^2\rho(1-\rho)\mathbf{s}^2$$

- **same** universal function

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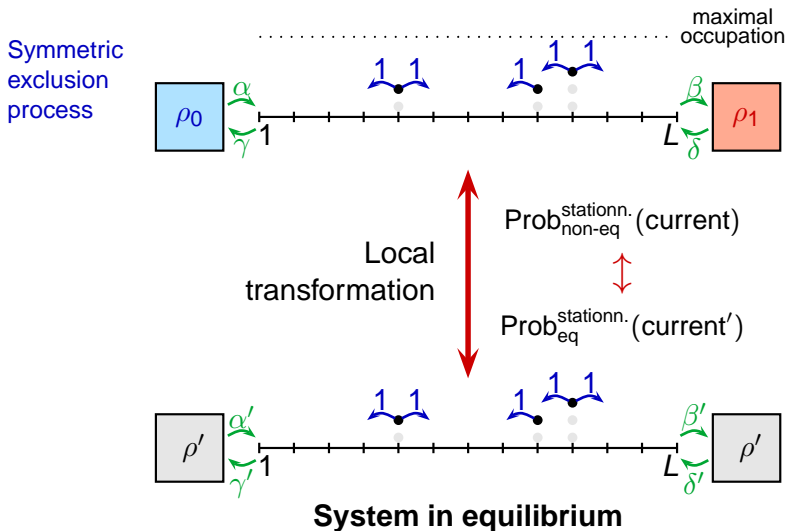
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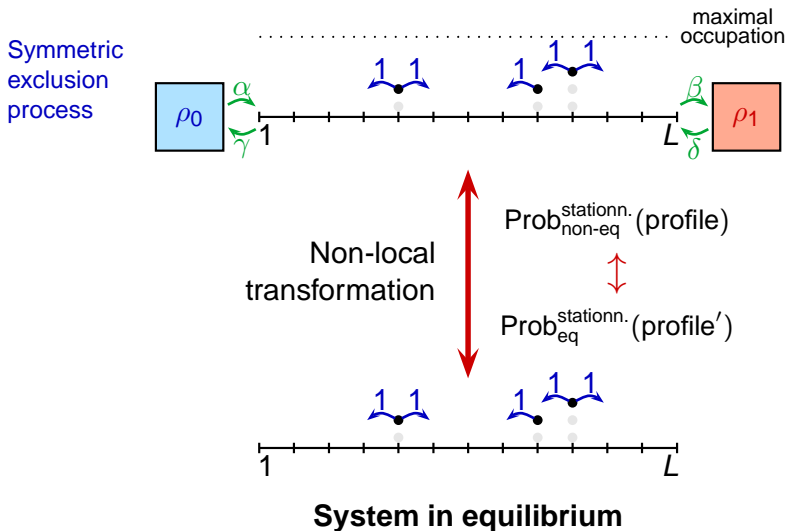
Correspondence between
the (Gaussian) integration of small fluctuations
AND
discreteness of Bethe root repartition.

More general?

Rk3: For the current

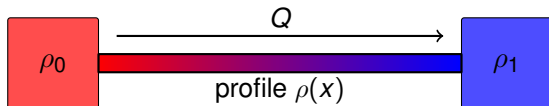
[Imparato, VL, van Wijland, **PRE** 80 011131]

Rk3: For the density profile [Tailleur, Kurchan, VL, JPA 41 505001]



Rk3: Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]

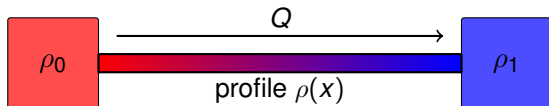


Boundary-driven transport model:

- long-range correlations
- breaking of time-reversal symmetry

Rk3: Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

- long-range correlations
- breaking of time-reversal symmetry

Non-local mapping to equilibrium:

- accounts for long-range correlations
- (density gradient)_{non-eq.} \longleftrightarrow (fixed density)_{eq.}
- yields $\text{Prob}[\rho(x)]$ through an extremalization principle

Summary

Approach:

- operator formalism
- large deviation function

Extensions:

- fluctuating hydrodynamics
- saddle-point method, instantons
- integration of fluctuations (dynamical phase transition)

Open questions:

- Eq \leftrightarrow non-eq mapping in higher dimensions?
- More generic systems of interacting particle?
- Link to other eq \leftrightarrow non-eq mappings?
- Crossover to KPZ? Universal fluctuations?

Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

For **periodic boundary conditions**

- large deviation function

$$\psi(\mathbf{s}) = \underbrace{L^{-1} \rho(1 - \rho) \mathbf{s}^2}_{\text{order 0}} + \underbrace{L^{-2} \mathcal{F}(u)}_{\text{finite-size}} \quad \text{with} \quad u = -\frac{1}{2} \rho(1 - \rho) \mathbf{s}^2$$

- universal function

$$\mathcal{F}(u) = \sum_{k \geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k) \Gamma(k+1)}$$

- scaling of cumulants of the total current Q_{tot}

$$\frac{1}{t} \langle Q_{\text{tot}}^2 \rangle \sim L$$

$$\frac{1}{t} \langle Q_{\text{tot}}^{2k} \rangle \sim L^{2k-2} \quad (k \geq 2)$$

Finite-size effects

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