

Noise and topology in driven systems—an application to interface dynamics

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Abstract

Motivated by a stochastic differential equation describing the dynamics of interfaces, we study the bifurcation behaviour of a more general class of such equations. These equations are characterized by a two-dimensional phase space (describing the position of the interface and an internal degree of freedom). The noise accounts for thermal fluctuations of such systems.

The models considered show a saddle-node bifurcation and have furthermore homoclinic orbits, i.e. orbits leaving an unstable fixed point and returning to it. Such systems display intermittent behaviour. The presence of noise combined with the topology of the phase space leads to unexpected behaviour as a function of the bifurcation parameter, i.e. of the driving force of the system. We explain this behaviour using saddle-point methods and considering global topological aspects of the problem. This then explains the non-monotonic force–velocity dependence of certain driven interfaces.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we consider a simplified, but general, description of driven dissipative systems described by two degrees of freedom in the presence of thermal noise. The theory applies to

systems with two phases separated by a rigid moving domain wall (DW) with an internal degree of freedom, but it also describes general stochastic differential equations having a homoclinic saddle bifurcation. We will study in detail the behaviour of such equations.

The paper is written with two audiences in mind; those interested and familiar with dynamical systems in the presence of noise—and those more interested in physical applications. A short account has been given in [11].

1.1. Physical motivation

A large variety of physical systems have interfaces separating different phases, with examples ranging from magnetic [4, 12, 14, 23] or ferroelectric [17, 18] DWs, to growth surfaces [2, 10] and contact lines [15]. The properties of an interface are well described at the macroscopic level by the competition between (i) the elasticity, which tends to minimize the interface length and (ii) the local potential, whose valleys and hills deform the interface so as to minimize its total energy.

The theory of disordered elastic systems [9, 8] allows one to determine their static and dynamical features (e.g. the roughness at equilibrium and the response to a field). Applying a force f , the interface can be driven to a non-equilibrium steady state. A crucial feature of the zero-temperature motion is the existence of a threshold force f_{crit} below which the system is pinned. The system undergoes a depinning transition at $f = f_{\text{crit}}$ and moves with a non-zero average velocity v for $f > f_{\text{crit}}$. Close to the transition the velocity $v \sim (f - f_{\text{crit}})^\beta$ is characterized by a depinning exponent β . In all these situations, the velocity is a *monotonic* function of the force (the more the interface is pulled, the faster it moves). Predictions of this theory are in very good agreement with experimental results, especially in the creep regime for interfaces in magnetic [12] or ferroelectric [18] films.

In spite of this success, there are situations where the disordered elastic theory does not apply: for instance, one basic assumption is that the bulk properties of the system are summarized by the position of the interface alone. Here, we study the case where the position of the interface is coupled to an internal degree of freedom and we will show how this coupling affects the motion of the interface. An example is provided by DWs in thin ferromagnetic films, where it is known that such an internal degree of freedom (*a phase*, to be detailed below) plays an important role.

We reproduce in figure 1 experimental measurements of the mean velocity. It is puzzling that the velocity is *not* a monotone function of the force.

The aim of this paper is to shed some light on this problem, by discussing a very simplified version of the system. We explain the general features of figure 1 by two ingredients: first, by observing that there is a change of the topology of a typical evolution as a function of the driving force, and second, by taking into account temperature. While we will work with a simplified potential, we will gain some quite general insights on this and related problems.

The experiments mentioned above come with physical models which describe the interaction between the phase φ and the position r of the wall. For the purposes of this paper, we will use the rigid wall approximation [3, 5, 13, 20–22]:

$$\begin{aligned} \alpha \partial_t r - \partial_t \varphi &= f - \cos(r) + \eta_1, \\ \alpha \partial_t \varphi + \partial_t r &= -\frac{1}{2} K_\perp \sin(2\varphi) + \eta_2. \end{aligned} \quad (1)$$

The external field $f - \cos(r)$ describes a constant ‘depinning’ (or ‘tilt’) force f and a ‘pinning’ force $-\cos(r)$ deriving from a periodic potential. The damping coefficient α accounts for Gilbert dissipation. The effective thermal noise is a white noise with correlations [5] $\langle \eta_i(t) \eta_j(t') \rangle = 2(\hbar N)^{-1} \alpha k_B T \delta(t' - t) \delta_{ij}$ where $N = 2\lambda A/a^3$ is the number of spins in the

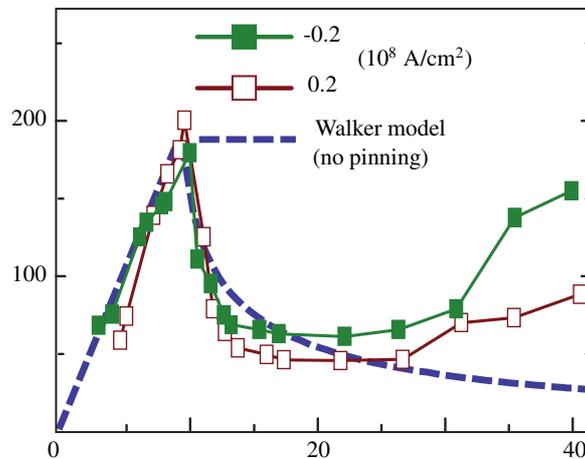


Figure 1. The experimental velocity (dots) of an interface in a narrow nanowire, driven by a small external current, adapted from [16]. The Walker model (represented by the dashed line), which discards the pinning potential, does not reproduce the experimental results. The horizontal axis is the force (magnetic field (Oe)), the vertical axis is velocity (m/s).

DW, where A is its cross-section, a the lattice spacing. Finally, K_{\perp} is the anisotropy constant of the ferromagnetic medium.

1.2. *Mathematical motivations*

The study of (1) reveals that the system has a saddle-node bifurcation at $f = f_S = 1$. Furthermore, for a range of fixed K_{\perp} one finds values of f for which the unstable manifold of the unstable fixed point is homoclinic⁵.

Saddle bifurcations have been discussed in many different contexts, and the influence of noise is well studied. Early papers are [6] and [1]. In those papers, the setup is that of intermittency in the presence of noise, with a discrete dynamical system of the form

$$x_{i+1} = x_i - \varepsilon + x_i^2 + \xi_i + h(x_i), \tag{2}$$

where $x_i \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$ is the bifurcation parameter, and ξ_i is some appropriate noise. The term $h(x)$ describes a function which, e.g. vanishes in the neighbourhood of $x = 0$ but is such that orbits must eventually return to a neighbourhood of 0. For this setting, under weak additional assumptions, one can study in quite some detail the invariant measure, and several salient features appear:

- Orbits stay a very long time close to $x = 0$ and do fast excursions away from the neighbourhood.
- When the parameter ε is positive, the deterministic system has a stable and an unstable fixed point (close to $x = \pm\sqrt{\varepsilon}$). The stochastic dynamics then helps the system to escape from the attracting fixed point (which is at $\sim -\sqrt{\varepsilon}$), but this may take a long time.

In this paper, we discuss a similar scenario, but with some new features: we consider the parametrization $f = 1 - \varepsilon^2$.

- (i) There are two equations (and they are differential equations rather than iterations), with a saddle-node bifurcation at the value of the bifurcation parameter $\varepsilon = \varepsilon_S = 0$.

⁵ More precisely, one side of the unstable manifold is homoclinic, while the other goes to a second (stable) fixed point (H_1 and S in figures 4 and 5).

- (ii) Close to $\varepsilon = 0$ there is an $\varepsilon_H > 0$ for which the unstable manifold (of the unstable fixed point) returns to the unstable fixed point⁶. We will be interested first in what happens for $\varepsilon \in (\varepsilon_S, \varepsilon_H)$.
- (iii) We then discuss how the topological type of the orbits can change when the phase space is a torus. This will lead to a non-monotonic mean sojourn time near the unstable fixed point.

The normal form of (1) is obtained by various rescalings, and a nonlinear coordinate transformation. The deterministic part is given by

$$\begin{aligned} dx &= (\varepsilon x + x^2) dt, \\ dy &= -y dt. \end{aligned}$$

While the deterministic part follows in a quite simple way, there is also a term appearing from the change-of-variables (the Itô term)⁷. This term takes the form $-\sigma^2 Q dt$ (in the x -component above, and a similar term for the y -component) with

$$Q = \frac{1 + \alpha^2}{8K_{\perp}\alpha^2} + \mathcal{O}(\varepsilon^{1/2}),$$

when $f = 1 - \varepsilon^2$. We will study (1) in the regime where $\varepsilon > 0$ and $\sigma^2 \ll \varepsilon$. (The simulations of figure 3 were done for $\varepsilon > 0.03$ and $\sigma^2 < 8 \cdot 10^{-7}$.)

Therefore, we continue the discussion of the local equation near the fixed point with the more easily tractable form

$$\begin{aligned} dx &= (\varepsilon x + x^2) dt + \sigma_x d\xi_x, \\ dy &= -y dt + \sigma_y d\xi_y, \end{aligned} \tag{3}$$

and omit the Itô term. Here, $d\xi_x$ and $d\xi_y$ describe the white noise, and the three parameters are $\varepsilon \geq 0$ and $\sigma_x \geq 0$, $\sigma_y \geq 0$. Adding a term h as in (2) one can achieve that globally the unstable manifold of $x = y = 0$ returns to $x = y = 0$ for some small $\varepsilon_H > 0$ when $\sigma = 0$. We will tacitly assume that such a term is present. The phase space of (1) is the torus $(r, \varphi) \in [0, 2\pi) \times [0, \pi)$ and the unstable fixed point is at $r = 0$, $\varphi = 0$. We will argue in section 5.2 that the term $\sigma_y d\xi_y$ can be omitted without changing the qualitative behaviour of the problem.

2. Results

We first present the results from a physicist's perspective:

In figure 2 we illustrate the first two findings which appear because a second field φ comes into play:

- The critical force, at which depinning initiates, moves from f_S , ($f_S = 1$), to a lower value f_H . Between f_H and f_S the system is bistable: the velocity is either 0 or strictly positive (see figure 2).
- The critical exponent of the velocity at depinning changes from $\frac{1}{2}$ to '+∞': the velocity grows like $v \sim 1/|\log(f - f_H)|$.

The physical picture behind the bistable regime is the following: the position r represents the position of a particle in a tilted periodic potential. For $f > f_S$ this potential presents no local minima and the velocity is positive. For $f < f_S$ there are local minima that cannot be

⁶ This is called a homoclinic connection.

⁷ We thank a referee for pointing out our oversight in not discussing this term in an earlier version of the paper.

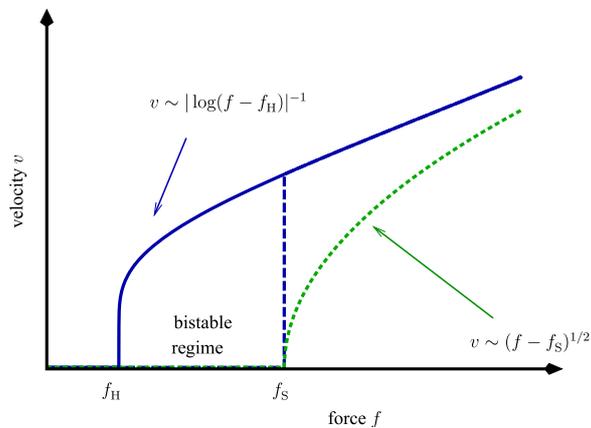


Figure 2. The two scenarios for the mean velocity: the green (dashed) curve shows v for the case of (4) where no internal degree of freedom is present, which corresponds to the motion of a particle on an inclined ‘washboard’. The blue (solid) curve shows the velocity v for the case of (1) where the DW position r is coupled to φ .

overcome in the absence of φ (this corresponds to the dashed curve of figure 2). The phase φ acts as an ‘energy store’ for the position r . If r starts close to a local minimum, dissipation makes it end at the minimum and the steady velocity is zero (this is the lower branch of the bistable regime in figure 2). On the other hand, if r starts far from a local minimum, the system reaches a stationary regime where φ helps r to ‘cross the energy barriers between successive minima, by periodically borrowing and giving ‘kinetic energy’ to r (this is the upper branch of figure 2).

Furthermore, if we introduce temperature, i.e. some external noise, then the force-velocity curve no longer presents any bistability (see figure 3). This leads to a third observation:

- The appearance of a third critical force, f_T . Note that for all (small) positive temperatures $T > 0$, the force-velocity curves actually cross at $f = f_T$, as illustrated in figure 3.

Our next result is a consequence of the periodicity of the rhs of (1) in φ . This periodicity is typical of DWs in magnetic systems. The DW position is generically coupled to an internal degree of freedom (for example a phase φ)⁸. In section 6 we will show how the periodicity influences the mean velocity

- The mean velocity of system (1) is a non-monotonic function with many maxima (depending on the values of α and K_\perp).

Remark 1. In early work [20], it was observed that, in the absence of pinning, i.e. because of the cosine in (1), $v(f)$ increases up to a characteristic force (called the Walker force) f_W above which the velocity *decreases* for a large range of $f > f_W$ see figure 1. What we show is that the pinning potential leads to a very different scenario.

3. A simple example

Leaving for a moment (1) aside, we study first a much easier problem to familiarize the reader with our approach. We consider the problem of depinning from a periodic potential, but

⁸ Although this coupling is well known in the magnetic DW community [13], it has to our knowledge always been discarded in interface physics.

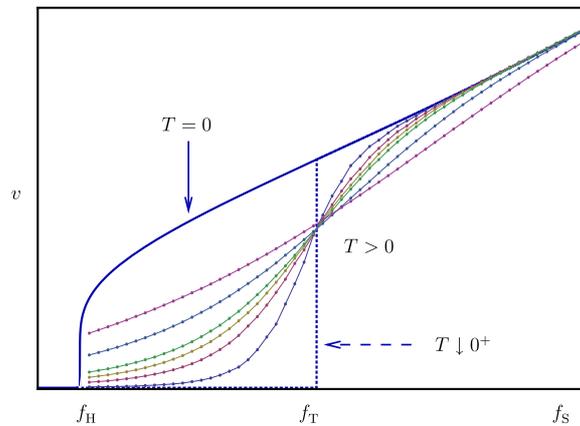


Figure 3. The velocity for (1) as a function of f and for several small values of the temperature T . Note that the curves cross at some value f_T . In the limit $T \downarrow 0$ the curves accumulate at f_T , while the deterministic equation (i.e. $T = 0$) leads to the blue curve. The parameters are $\varepsilon^2 \in \{0.03, 0.1\}$, $\sigma \in \{0.0001, 0.0009\}$, $\alpha = 1/2$ and $K_{\perp} = 6$. The integrator was Euler–Maruyama, with a time step of 0.0003. We averaged over 512 samples.

without the phase φ . The common underlying ingredient of such systems is the ‘pulling’ of an interface by a force f . As the easiest example, we can consider the case of an ‘inclined washboard’:

$$\partial_t r = f - \cos(r), \tag{4}$$

where f is the constant force and $r = r(t) \in \mathbb{R}$ is the position of the DW at time t . Clearly, the rhs of (4) can vanish only if $|f| \leq 1$, and in that case every initial condition $r_0 = r(0)$ will, as time evolves, converge to one of the values $r_* = \arccos(f) + 2\pi n$, with n any integer. In this case, we say that the potential is *pinning*. On the other hand, when $|f| > 1$, there is no fixed point for (4) and $r(t)$ will increase or decrease indefinitely. In fact, one can check that, for $f > 1$ and $r(0) = 0$ the solution of (4) is

$$r(t) = 2 \arctan \left(\frac{\tan \left(\frac{1}{2} t \sqrt{f^2 - 1} \right) \sqrt{f^2 - 1}}{f + 1} \right),$$

whose derivative is a periodic function of t with period $p = 2\pi/\sqrt{f^2 - 1}$. Therefore, $r(p) - r(0) = 2\pi$, and the limit is

$$\lim_{t \rightarrow \infty} \frac{r(t)}{t} = \frac{\sqrt{f^2 - 1}}{2\pi} = \mathcal{O}(\sqrt{f - 1}).$$

In other words, the mean displacement, which we call the *velocity* v , is given by $v(f) = \sqrt{f^2 - 1}/(2\pi)$. Thus, for the simple case of (4) the well-known result is that near the depinning transition, the velocity grows like $\sqrt{f_S - f}$ where $f_S = 1$ in our simple example. Furthermore, the velocity is obviously a monotone function of f : the harder one pulls, the faster one advances.

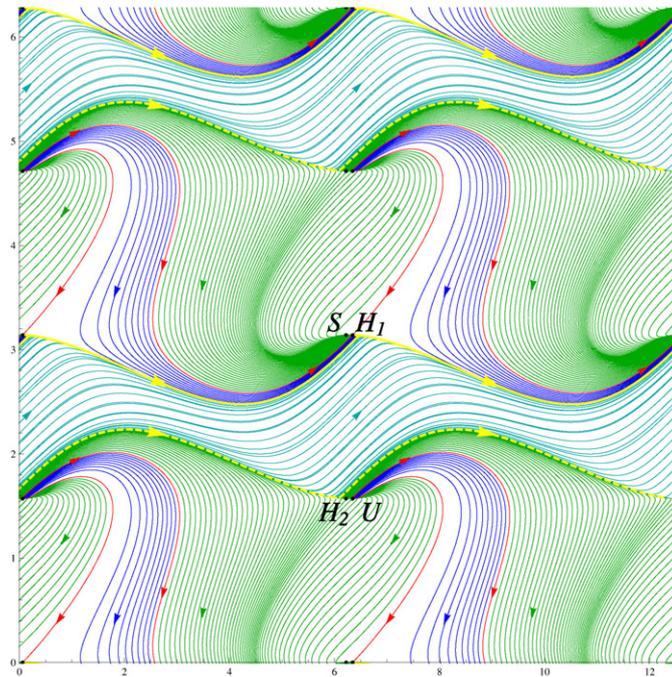


Figure 4. Phase portrait of the evolution (1) in the (r, φ) plane, for $\varepsilon_S = 0 < \varepsilon < \varepsilon_H$. Boundaries between different regions are in red. The attracting limit cycle is shown as a yellow dashed line, while the repulsive limit cycle is shown as a yellow solid line. In green, the basin of attraction of the stable fixed point S ; in blue, the basin of attraction of the stable limit cycle, either from below (light blue) or from above (dark blue).

4. The coordinates of the problem at zero temperature

We now study the special case of (1) when the noise terms are absent

$$\begin{aligned} \alpha \partial_t r - \partial_t \varphi &= f - V'(r), \\ \alpha \partial_t \varphi + \partial_t r &= -\frac{1}{2} K_{\perp} \sin(2\varphi), \end{aligned} \tag{5}$$

where $V'(r) = \cos(r)$.

When K_{\perp} is very large, φ will be very close to $0 \bmod \pi$, and then the system reduces to the washboard model (4). However, for smaller K_{\perp} , the phase φ matters and this is the case we want to study now. A redefinition of $f = 1 - \varepsilon^2$ brings problem (5) to the more convenient form

$$\begin{aligned} \alpha \partial_t r - \partial_t \varphi &= -\varepsilon^2 + (1 - \cos(r)), \\ \alpha \partial_t \varphi + \partial_t r &= -\frac{1}{2} K_{\perp} \sin(2\varphi). \end{aligned} \tag{6}$$

The phase space of this equation is the torus $(r, \varphi) \in [0, 2\pi) \times [0, \pi)$. For the following discussion the reader is referred to figure 4 where the torus is drawn in the plane with the horizontal axis corresponding to r and the vertical corresponding to φ . A three-dimensional rendering is shown in figure 5.

It is easily verified that (6) is invariant under the symmetry: $r \rightarrow -r, \varphi \rightarrow -\varphi + \pi/2, t \rightarrow -t$. This makes the phase space centrally symmetric, but we will not make use of this property in the analysis.

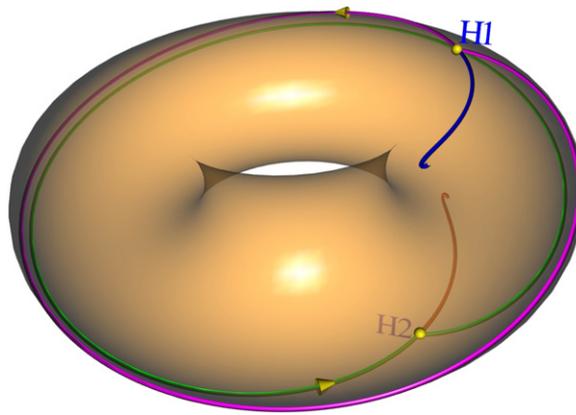


Figure 5. The unstable manifold (purple) of the (yellow) hyperbolic fixed point H_1 winds around the torus once (counterclockwise) and ends at the fixed point H_1 . In green (behind) the same for the other fixed point H_2 . The stable fixed point is at the end of the blue 'tail', and the unstable at the end of the orange tail.

We will consider only values of $\varepsilon^2 \geq 0$, $K_{\perp} > 0$. For the simulations, we took $\alpha = \frac{1}{2}$. Under these assumptions, the *local* structure of this equation is characterized by 4 fixed points of the form $(0, \pm r_{\varepsilon})$ and $(\pi/2, \pm r_{\varepsilon})$, where

$$r_{\varepsilon} = \arccos(1 - \varepsilon^2).$$

The stability of the four fixed points is as follows (for $\varepsilon > 0$):

- $H_1 = (0, r_{\varepsilon})$ and $H_2 = (\pi/2, -r_{\varepsilon})$ are hyperbolic (with one stable and one unstable direction),
- $S = (0, -r_{\varepsilon})$ is stable,
- $U = (\pi/2, r_{\varepsilon})$ is unstable.

For $\varepsilon = 0$ we have $r_{\varepsilon} = 0$ and the corresponding pairs of fixed points collide, leading to a single fixed point with one direction stable, and the other stable-unstable. Thus, at $\varepsilon = 0$ the fixed points S and H_1 (respectively U and H_2) *collide*; we are in the presence of a typical *saddle-node bifurcation*.

5. General discussion for the case of non-zero temperature

Apart from its interest as a physics problem, the equations under study are a nice example of the interplay of homoclinic orbits, collision of a saddle-node, and the influence of noise. While any combination of two of the three phenomena is amply discussed in the literature [19], as far as we know, the combination of all three seems to be new. In particular, as we shall show, the system will have a 'phase transition' as the noise goes to zero, which occurs neither at the homoclinic point, nor at the collision of the saddle-node, but at a well-defined intermediate point. The present section will derive this in a general form.

5.1. The one-variable case

In very early work, Riskin [19], considered the problem

$$\partial_t^2 r = -\gamma \partial_t r - \varepsilon + (1 - \cos(r)).$$

If we write it as a first order system, we have

$$\begin{aligned}\partial_t x &= v, \\ \partial_t v &= -\gamma v - \varepsilon + (1 - \cos(x)).\end{aligned}$$

The phase space for this system is $(x, v) \in [0, 2\pi) \times \mathbb{R}$. There are now only two fixed points: $v = 0, x = x_* \equiv \pm \arccos(1 - \varepsilon)$. So, the system is really quite different from our model. However, two of its main features remain and they can be discussed in the spirit of (1): locally, there are two fixed points: one is stable $(v, x) = (0, x_*)$ and the other $(0, -x_*)$ is hyperbolic. Again, for $\varepsilon = 0$ there is collision of the two fixed points (a saddle-node bifurcation). On the other hand, there is a value ε_H of ε (not the same as in our model) depending on γ for which we have a homoclinic connection.

5.2. The 2-variable case

We consider again (1), but change coordinates immediately to a normal form. Furthermore, for the purpose of the discussion in this section, it is irrelevant that the natural phase space is the torus. In fact, it suffices to consider a local coordinate system near the saddle-node. The global aspects only have to do with the ‘rejection’ [6].

In a local coordinate system where the hyperbolic fixed point H_1 is at the origin, up to terms of higher order, and neglecting the Itô term, as discussed in section 1.2, the system can be written in the form

$$\begin{aligned}dx &= (\varepsilon x + x^2) dt + \sigma d\xi, \\ dy &= -y dt + \sigma_2 d\xi_2.\end{aligned}\tag{7}$$

Here, $d\xi$ and $d\xi_2$ describe white noise, and the three parameters are $\varepsilon \geq 0, \sigma \geq 0$, and $\sigma_2 \geq 0$.

We will omit the noise term in the (stable) y direction because it would only induce a fluctuation in the ‘arrival time’, but has no influence on the escape rate to the basin of attraction of the fixed point $(-\varepsilon, 0)$. However, the noise in the x direction is essential for our discussion.

There is one more, crucial, assumption: for some $\varepsilon_H > 0$ (when $\sigma = 0$) the unstable manifold of the fixed point $(x, y) = (0, 0)$ (in the positive direction) is homoclinic, that is, it returns to $(0, 0)$. Furthermore, for $\varepsilon < \varepsilon_H$ the unstable manifold is moved to the right (positive x). See figure 6. We also assume that this unstable manifold is transversally stable, that is, nearby orbits are attracted to it, as illustrated in figure 6. Such behaviour can be obtained if in (7) we add some nonlinear terms which bend the unstable manifold of $(0, 0)$ as shown in figure 6. We assume in the following that (7) has been modified accordingly, without changing the vector field near $(0, 0)$.

Proposition 5.1. *Under the above assumptions, there is a constant $A > 0$ such that the mean velocity of system (7) has a phase transition at a point ε_T and, for small ε and large ε^3/σ^2 , this transition happens at ε_T close to the solution of*

$$\varepsilon - 6(A \cdot (\varepsilon_H/\varepsilon - 1))^2 = 0.$$

(This solution lies in the interval $(\varepsilon_S = 0, \varepsilon_H)$.)

The essential thing here is that $\varepsilon_H > 0$. When $\sigma > 0$ the following happens: if we start at some point of the unstable manifold, and evolve with the noisy evolution, the orbits come back, for ε between $\varepsilon_S = 0$ and ε_H , as a basically Gaussian distribution around the unstable manifold, see figure 7.

At this point, an intriguing competition between two phenomena occurs. On one hand, because $\sigma > 0$, some orbits (those on the ‘inside’ of the homoclinic loop in figure 7) are

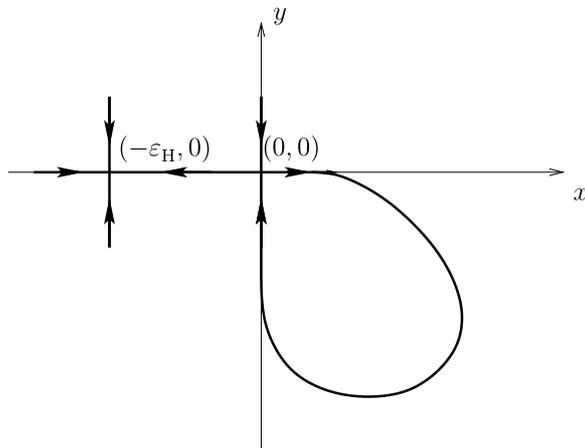


Figure 6. The phase portrait at $\varepsilon = \varepsilon_H$.

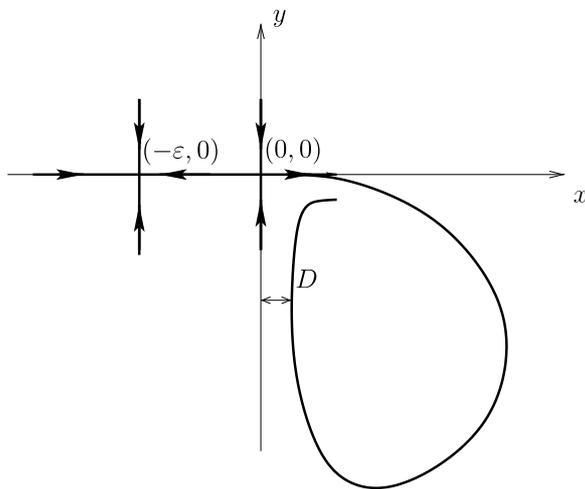


Figure 7. The phase portrait at $\varepsilon < \varepsilon_H$. The unstable manifold of $(0, 0)$ has moved to the right by a distance $D \sim A \cdot (\varepsilon_H - \varepsilon)$.

accelerated by the noise, since they avoid the close passage by the fixed point $(0, 0)$. On the other hand, those which return to $(0, 0)$ on the side of $x < 0$ fall into the basin of attraction of the fixed point $(-\varepsilon, 0)$ and they will need a long time to escape from that basin. This phenomenon has been studied long ago under the name of ‘intermittency in the presence of noise’ [6]. In that case, it was always (rightly) assumed that the reinjection density is close to uniform across the basin. In the case at hand, the novel problem is that the probability to fall into the basin of attraction of the point $(-\varepsilon, 0)$ decreases as ε decreases from ε_H . This is because the center of the probability distribution of orbits moves away from the basin as ε decreases, see figure 8.

To quantify this phenomenon, we assume that to lowest order, the unstable manifold is moved by an amount $A \cdot (\varepsilon_H - \varepsilon)$ in the positive direction, i.e. $A > 0$. The potential along the x -axis is shown in figure 9.

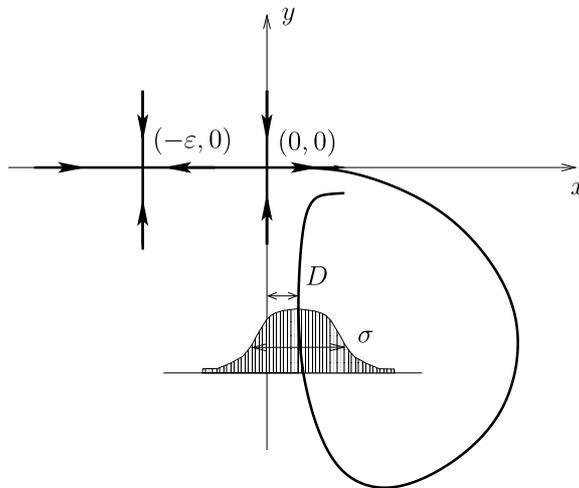


Figure 8. The same phase portrait as in figure 7. Superposed is the (Gaussian) distribution of noisy orbits returning along the unstable manifold of $(0, 0)$. Note that the relation between D and the width σ of the distribution determines how frequently a noisy orbit will fall onto the stable (left) side of the y -axis.

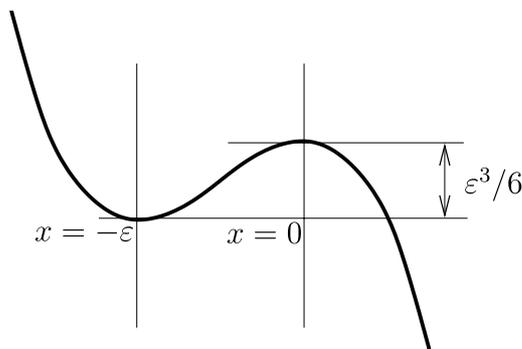


Figure 9. The typical shape of the effective local potential $V(x) = -\epsilon x^2/2 - x^3/3$ near $x = 0$. Note that the depth of the potential and its width depend on ϵ .

We next ask, for a fixed $\epsilon \in (0, \epsilon_H)$ and a fixed $x \in [0, \epsilon]$ how long the stochastic process (7), starting at x , needs to escape to the right (to $+\infty$). We will neglect the y coordinate in this estimate. As is well known, and for example done in detail in section 3 of [6], see also [7], this time is given by Green’s function of the differential operator G , (7):

$$G = \frac{\sigma^2}{2} \frac{d^2}{dx^2} + (\epsilon x + x^2) \frac{d}{dx}, \tag{8}$$

with Dirichlet boundary condition at $x = +\infty$. The expected time $\tau(x)$ to escape from x is then given by

$$\tau(x) = \frac{2}{\sigma^2} \int_x^\infty dz e^{-h(z)} \int_{-\infty}^z dw e^{h(w)}, \tag{9}$$

with the potential $h = -V$ given by

$$h(z) = \frac{2}{\sigma^2} \left(\frac{\varepsilon z^2}{2} + \frac{z^3}{3} \right).$$

The integral (9) can be estimated as in [6]. First one changes variables to $u = z + w$ and $v = z - w$ and finds

$$\tau(-\infty) = \frac{2}{\sigma^2} \int_{-\infty}^{\infty} du \int_0^{\infty} dv \exp \left(-\frac{2}{\sigma^2} \left(\frac{1}{12} v^3 + \left(\frac{\varepsilon u}{2} + \frac{\varepsilon u^2}{4} \right) v \right) \right). \quad (10)$$

(Pushing the integration limit to $x = -\infty$ is justified by the fact that anyway, most of the time is spent near $x = -\varepsilon$.) The u integration can be done explicitly and leads to

$$\tau(-\infty) = \frac{2}{\sigma^2} \int_0^{\infty} dv \left(\frac{\pi}{v} \right)^{1/2} \exp \left(-\frac{2}{\sigma^2} \left(\frac{v^3}{12} - \frac{\varepsilon^2 v}{4} \right) \right). \quad (11)$$

We rescale by $v = \varepsilon w$ and thus find

$$\tau(-\infty) = \frac{2\varepsilon^{1/2}}{\sigma^2} \int_0^{\infty} dw \left(\frac{\pi}{w} \right)^{1/2} \exp \left(-\frac{2\varepsilon^3}{\sigma^2} \left(\frac{w^3}{12} - \frac{w}{4} \right) \right). \quad (12)$$

Using the saddle-point approximation (the critical point is at $w = 1$) this integral behaves, for large ε^3/σ^2 , and neglecting the prefactor in front of the exponential as

$$\tau(-\infty) \sim \exp \left(\frac{\varepsilon^3}{3\sigma^2} \right).$$

On the other hand, as illustrated in figure 8, the probability to reach a point $x < 0$ is proportional to $\exp(-\text{const} \cdot (|x| + F)^2/\sigma^2)$, where the constant F depends on certain global aspects of the problem, such as the length of the (almost) homoclinic loop. This just estimates how much probability leaks to the ‘wrong’, i.e. left side of the unstable manifold of $(0, 0)$. We will continue the discussion by assuming all the constants to be 1. A rescaling of the variables would eliminate an arbitrary constant anyway. In particular, the average time to leave the trap (say, between $x = -\varepsilon$ and $x = 0$) is then given approximately by

$$\tau_{\text{escape}} \sim \exp \left(\frac{\varepsilon^3}{3\sigma^2} - \frac{2}{\sigma^2} (A \cdot (\varepsilon_{\text{H}} - \varepsilon))^2 \right). \quad (13)$$

Here, we have used that, to lowest order, $D = A \cdot (\varepsilon_{\text{H}} - \varepsilon)$.

Consider now the polynomial in (13). It can be written as

$$\frac{\varepsilon^2}{3\sigma^2} (\varepsilon - 6(A \cdot (\varepsilon_{\text{H}}/\varepsilon - 1))^2).$$

For fixed ε_{H} , this polynomial has exactly one real root $\varepsilon_{\text{T}} = \varepsilon_{\text{T}}(A, \varepsilon_{\text{H}})$ which lies in $(0, \varepsilon_{\text{H}})$. This is the point where the behaviour will switch over. It is the point which corresponds to f_{T} in figure 3.

6. Global topological aspects

After having neglected the torus structure of the problem, we reinstate it in the current section. If we want to perform a global study of the system in the parameters ε and K_{\perp} we have to take into account that the phase space of (6) (or (1)) is a torus. Thus there can (and do) exist several topologically different ways in which a homoclinic orbit can form. They can be indexed by two (non-negative) integers ϱ and ψ which count how many turns of 2π the variable r respectively 2φ will undergo as one moves from the fixed point H_1 to reach it again through the homoclinic loop, and denote by $\mathcal{W}(\varrho, \psi)$ the index of the orbit.

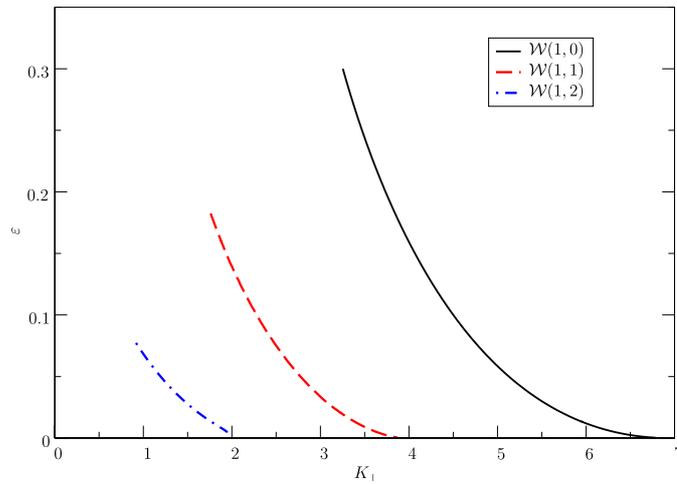


Figure 10. The locus of some homoclinic connections in the ε, K_{\perp} plane. Shown are 3 such curves with winding number in r equal to 1, and in φ equal to 0, 1, 2. There are infinitely many such curves.

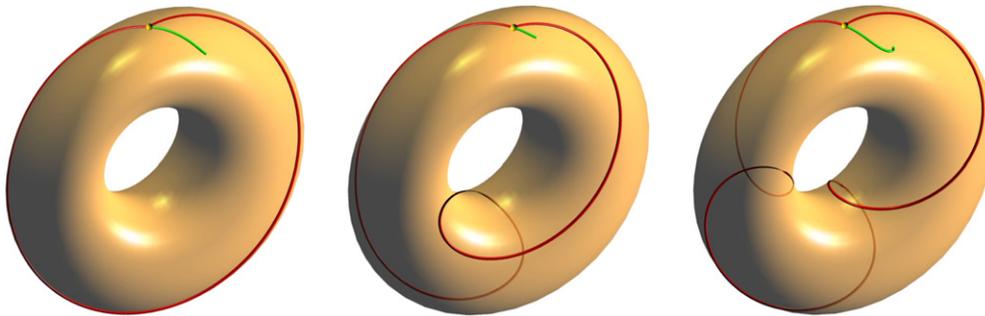


Figure 11. From left to right: 3 homoclinic orbits, of type $\mathcal{W}(1, 0)$, $\mathcal{W}(1, 1)$, and $\mathcal{W}(1, 2)$, respectively.

In the space of ε and K_{\perp} , the picture which emerges numerically is shown in figure 10. For each of the curves in figure 10 we show one example in figure 11.

It is now easy to explain the non-monotonicity of the mean velocity for (1), as illustrated in figure 12. Fixing a K_{\perp} (say $K_{\perp} = 3.5$ in figure 10) and varying the pulling force $f = 1 - \varepsilon$ for ε from 0 to 0.3 we first cross the $\mathcal{W}(1, 1)$ curve and then the $\mathcal{W}(1, 0)$ curve. This leads to figure 12. Note that, in accordance with the theory of section 5, as a function of the noise (temperature), both bumps are filled with details which look as in figure 3. In particular, there will be a special bifurcation point of the form f_T for both of them.

In figure 12, we illustrate the cases $\mathcal{W}(1, 0)$ and $\mathcal{W}(1, 1)$. The reader should notice that for every pair (ϱ, ψ) which is realized, for fixed K_{\perp} , there will be a window $\mathcal{W}_{\varrho, \psi}$ of values of f around $f_H(\varrho, \psi)$ which is like the case we discussed in detail. *Mutatis mutandis* our analysis will apply immediately to all these cases, as soon as the general hypotheses (about the transverse stability of the homoclinic orbit and the motion of the return as a function of f) are satisfied.

The physical interpretation of the non-monotonic behaviour of the velocity is the following. We have seen previously (see section 2) that φ ‘helps’ r to cross the barriers between the local

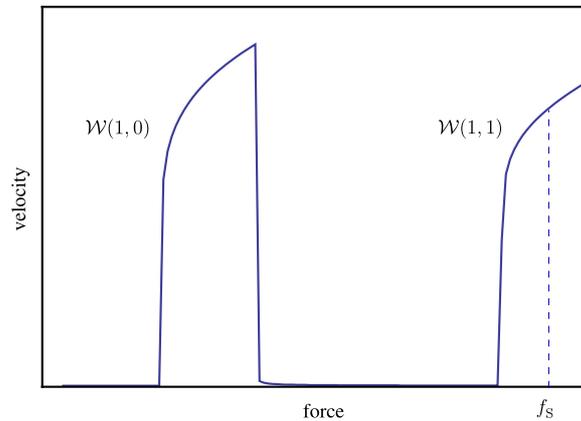


Figure 12. A schematic illustration of the velocity as a function of force. First the topological type $\mathcal{W}(1, 0)$ leads to a monotone increase of the velocity. But at some point, the winding number for φ changes, and the velocity drops to 0. Several other winding numbers, not shown, could occur before the topological case $\mathcal{W}(1, 1)$ sets in.

minima of the potential where it lives. Doing so, φ oscillates in its own local minimum (this corresponds to the first bump in figure 12). However, for larger values of the force, φ will itself cross the barriers of its potential and dissipate so much energy that it cannot help r anymore (in the phase space picture of figure 4, this corresponds to a collision between the attractive and repulsive limit cycle). There is a whole regime of force where no limit cycle exists (this is the flat region between the two bumps in figure 12). It is only for larger values of f that a stationary regime appears where both φ and r can cooperate and display non-zero mean velocity (this corresponds to the second bump in figure 12).

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