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Current distribution in systems with anomalous diffusion: renormalization group approach

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Abstract
We investigate the asymptotic properties of the large deviation function of the integrated particle current in systems, in or out of thermal equilibrium, whose dynamics exhibits anomalous diffusion. The physical systems covered by our study include mutually repelling particles with a drift, a driven lattice gas displaying a continuous nonequilibrium phase transition and particles diffusing in an anisotropic random advective field. It is exemplified how renormalization group techniques allow for a systematic determination of power laws in the corresponding current large deviation functions. We show that the latter are governed by known universal scaling exponents, specifically the anomalous dimension of the noise correlators.

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1. Introduction

1.1. Motivations

Recently an important concept for the analysis of systems driven out of thermal equilibrium, namely that of *large deviations* in the associated probability distributions, has again found considerable attention, at both the experimental and theoretical levels. Mathematically, large deviation functions have been defined and introduced long ago, but only in the early 1970s were they shown to play a central role in characterizing the properties of non-equilibrium steady state (NESS) measures (see [1] for a review). For some time this idea remained mostly confined within the mathematical community until it was exploited in physics at the theoretical level by Evans, Cohen and Morris who showed numerically that the temporal large deviation function of a certain observable, namely the shear stress, in the NESS of viscous gases under shear displayed a surprisingly simple symmetry property [2]. This symmetry property, now known as the *fluctuation relation*, was then formalized into a theorem by Gallavotti and Cohen [3]. Over the past few years, a remarkable amount of research has been devoted to the investigation of the fluctuation relation in detail, with the goal to explicitly determine the large deviation function at the core of the theorem's formulation [4–7]. It must be emphasized that at about the same period, experimentalists also started to employ, as a tool for analysing their data, the concept of large deviations [8]. Their initial interest was grounded in the belief that, much in the same way as the intensive free energy per degree of freedom constitutes a central quantity if one wishes to relate two systems in thermal equilibrium, large deviation functions might similarly allow for a comparison of NESS *dynamical* properties, yet to be applicable to systems both in and out of equilibrium. Indeed, it has been a highly desirable, but as yet largely elusive goal of non-equilibrium statistical mechanics to be able to classify dynamical systems (at least in a steady state) according to a restricted number of (macro- or mesoscopic) criteria. In particular, it was expected that large deviation functions of observables such as energy flow, particle current (or more generally entropy flow) would display a sufficient degree of universality to render this approach fruitful (see [4–14]).

On the theoretical side, a series of recent results however lends support to such aspirations. By studying systems of interacting and diffusing particles governed by some Markovian dynamics, subjected to an external chemical potential gradient, Bodineau and Derrida [9–11], as well as Bertini, De Sole, Gabrielli, Jona-Lasinio and Landim [12–14], found a general strategy to completely determine the large deviation of the particle current flowing through the system. Note, however, that in this setting the applied field (and the related particle current) become vanishingly small as the system size is increased. Then the only two ingredients entering the final expression for the large deviation function are the diffusion constant and the equilibrium compressibility, which is an astonishingly simple result: the large deviations of the current are fully characterized by properties of the system at equilibrium or in the linear response regime.

There are two restrictions, however, that apply to this series of recent advances. Namely, first they all concern systems exhibiting *normal* diffusive dynamics. In physical terms, this means that the continuum limit can be taken by scaling time according to $\tau \sim a^2$ proportional to a microscopic length scale a squared (often referred to as the hydrodynamic limit). There are of course numerous situations in nature where diffusion becomes *anomalous*, which means that the time and length scales appropriate for taking the continuum limit are related by $\tau \sim a^z$ with $z \neq 2$. The purpose of the present work is to investigate the particle current large deviation functions in such physical systems governed by anomalous diffusion; specifically, we shall address the following models: (i) mutually excluding particles in a driving 'electric' field [15, 16]; here $z = 3/2$ in one space dimension, which is equivalent to the scaling for the noisy Burgers equation [17], in two dimensions one finds $z_\parallel = 2$ with logarithmic corrections along the drive direction, while normal diffusive behaviour ensues transverse to it, $z_\perp = 2$; (ii) critical driven diffusive systems (driven model B) [18, 19], for which one similarly obtains $z_\parallel = 12/(11-d)$ in dimensions $d < 5$, with again logarithmic corrections at the upper critical dimension $d_c = 5$, and $z_\perp = 2$; and (iii) particle diffusion in a quenched random velocity field, where according to the type of disorder, both super- and sub-diffusive behaviour may occur [20, 21]. Based on a field theory representation of the nonlinear Langevin-type stochastic differential equations representing the above processes [22–25], renormalization group (RG) methods will prove instrumental in extracting the appropriate asymptotic scaling variables in the long-time and large-system limit, and will be shown to provide the relevant tool for taking the proper continuum limit [26, 27].

The second restriction for most recent investigations is that these studies apply only to systems that are weakly out of equilibrium, since, as already mentioned before, the applied external field scales as the reciprocal of the system size. To the best of our knowledge, it is only for the one-dimensional exclusion process with periodic boundary conditions, with [28] or without a bias [7] in particle propagation, that the current fluctuations have been accessed analytically to date. The efforts of the present paper therefore bear on systems with anomalous diffusion in dimensions $d \geqslant 1$, some of which driven far away from thermal equilibrium.

In the course of this work we shall investigate the properties of the current large deviation in three different systems, which we define succinctly in the following subsection. In sections 2, 3 and 4 we present the RG analysis and our results for each of these model systems. Section 2 itself will also expand on the field-theoretic methods upon which we shall rely, whereas most of the technical details will be skipped in the subsequent sections 3 and 4. Our conclusions are gathered in section 5.

1.2. Presentation of the model systems

We will first investigate the properties of particles (without source nor sink) subjected to a driving force along one spatial direction (denoted '$\parallel$' in the following), whose local density $\rho(\mathbf{x}, t)$ evolves according to the continuity equation $\partial_t \rho(\mathbf{x}, t) = -\boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{x}, t)$, with a particle current $\mathbf{j} = -D\boldsymbol{\nabla}\rho + \rho\mathbf{u}(\rho) + \boldsymbol{\eta}$ constructed from phenomenological considerations. The first term here is a Fickian diffusion current, the second contribution originates from the driving force and describes a density-dependent velocity field $\mathbf{u}(\rho)$ along the drive, and finally $\boldsymbol{\eta}$ represents Gaussian white noise which accounts for fast degrees of freedom not taken into account explicitly in our mesoscopic description. The former continuum Langevin description arises for instance from a discrete lattice gas description, which we shall use in the following to determine the particular form of the velocity field $\mathbf{u}(\rho)$. If we transform from a lattice occupation number representation ($n_i = 0, 1$) of the discretized system (with lattice sites labelled by the index i) to Ising variables $\phi_i = n_i - 1/2 = \mp 1/2$, particle–hole ($Z_2$) symmetry dictates that to lowest order in the coarse-grained local density field $\phi(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho_0$ we have $u_\parallel(\phi) = \varepsilon(1 - \phi^2) + \cdots$. Denoting by $\mathbf{j}_0$ the average current, renaming $\varepsilon = Dg/2$, and allowing for a different effective diffusivity λD along the drive direction, we thus arrive at the following expressions for the transverse and longitudinal (component along the drive) currents

$$\mathbf{j}_\perp - \mathbf{j}_{\perp 0} = -D\boldsymbol{\nabla}_\perp \phi + \boldsymbol{\eta}_\perp, \tag{1}$$

$$j_\parallel - j_{\parallel 0} = -\lambda D \nabla_\parallel \phi - \frac{Dg}{2}\phi^2 + \eta_\parallel. \tag{2}$$

The parameter g accounts for the interaction of particles along the driving field. After rescaling such that the transverse current sector is effectively in equilibrium, the noise correlations read

$$\langle \eta_{\perp i}(\mathbf{x}, t)\eta_{\perp j}(\mathbf{y}, t')\rangle = 2D\delta_{ij}\delta^{(d)}(\mathbf{x} - \mathbf{y})\delta(t - t'), \tag{3}$$

$$\langle \eta_\parallel(\mathbf{x}, t)\eta_\parallel(\mathbf{y}, t')\rangle = 2D\sigma\delta^{(d)}(\mathbf{x} - \mathbf{y})\delta(t - t'). \tag{4}$$

Collecting everything into a single stochastic Langevin equation for the density field ϕ, we at last arrive at

$$\partial_t \phi = D\left(\nabla_\perp^2 + \lambda\nabla_\parallel^2\right)\phi + \frac{Dg}{2}\nabla_\parallel \phi^2 + \zeta, \tag{5}$$

$$\langle \zeta(\mathbf{x}, t)\zeta(\mathbf{y}, t')\rangle = -2D\left(\nabla_\perp^2 + \sigma\nabla_\parallel^2\right)\delta^{(d)}(\mathbf{x} - \mathbf{y})\delta(t - t'), \tag{6}$$

where $\zeta = -\nabla \cdot \boldsymbol{\eta}$. Further motivations for this phenomenological approach, and a detailed study of the scaling properties of this model can be found in [15, 16, 18, 27, 29]. We emphasize that in the presence of a drive, and with periodic boundary conditions, (5) and (6) in general describe a non-equilibrium system. In one dimension, they reduce to the noisy Burgers equation [17].

For our second example, we turn to the standard model of a driven diffusive system that exhibits a continuous non-equilibrium phase transition in its stationary state [18, 19]. In this case, the transverse current looks like the one for the time-dependent Ginzburg–Landau model with conserved order parameter,

$$\mathbf{j}_\perp - \mathbf{j}_{\perp 0} = -D\nabla_\perp \left(r - \nabla_\perp^2 + \frac{u}{6}\phi^2 \right) \phi + \boldsymbol{\eta}_\perp, \tag{7}$$

while the current component along the drive direction is still given by (2), and the noise correlations by (3) and (4). For the field ϕ, this yields the Langevin equation

$$\partial_t \phi = D\nabla_\perp^2 \left(r - \nabla_\perp^2 + \frac{u}{6}\phi^2 \right) \phi + \lambda D \nabla_\parallel^2 \phi + \frac{Dg}{2} \nabla_\parallel \phi^2 + \zeta, \tag{8}$$

with the noise correlator (6). The system now has a critical point at $r = 0$ (within the mean-field approximation), which in the absence of the drive $g = 0$ just describes the second-order phase transition in the Ising model with Kawasaki dynamics (model B). In the presence of the drive ($g \neq 0$), however, the universality class changes; in fact, u becomes irrelevant for the critical properties of the driven model B, which are entirely governed by the nonlinear driving term [18].

Finally, we consider non-interacting particles diffusing in a random velocity field, as defined by Honkonen [20]. At the mesoscopic level, the system is described over a d-dimensional continuum by a local density fluctuation $\phi(\mathbf{x}, t)$ subjected to a random field $\psi(\mathbf{x}_\perp)$ directed along $\mathbf{e}_\parallel$. One primary feature of the model is that the amplitude of the field depends only on the transverse coordinates $\mathbf{x}_\perp$. The density field $\phi(\mathbf{x}, t)$ evolves according to the Langevin equation $\partial_t \phi + \nabla \cdot \mathbf{j} = 0$ with a current $\mathbf{j}$ that accounts for the random field $\psi(\mathbf{x}_\perp)$ solely in the parallel direction:

$$\mathbf{j}_\perp = -D_\perp \nabla_\perp \phi + \boldsymbol{\eta}_\perp, \tag{9}$$

$$j_\parallel = -D_\parallel \nabla_\parallel \phi - \psi\phi + \eta_\parallel. \tag{10}$$

The thermal diffusion of particles is encoded by the white noise $\boldsymbol{\eta}$ which enters the expression for the current. Its correlations ensure that both the transverse and longitudinal sectors are in equilibrium,

$$\langle \eta_{\perp,i}(\mathbf{x}, t)\eta_{\perp,j}(\mathbf{x}', t')\rangle = 2D_\perp \delta_{ij}\delta(\mathbf{x}' - \mathbf{x})\delta(t' - t), \tag{11}$$

$$\langle \eta_\parallel(\mathbf{x}, t)\eta_\parallel(\mathbf{x}', t')\rangle = 2D_\parallel \delta(\mathbf{x}' - \mathbf{x})\delta(t' - t). \tag{12}$$

Depending on the spatial correlations of the random field $\psi(\mathbf{x}_\perp)\mathbf{e}_\parallel$, particles can behave either sub- or superdiffusively [20, 21]. We restrict our analysis to the case of a spatially uncorrelated random field with Gaussian distribution, whose correlations are described by a constant amplitude λ:

$$\langle \psi(\mathbf{x}_\perp, t)\psi(\mathbf{x}'_\perp, t')\rangle = \lambda \delta(\mathbf{x}'_\perp - \mathbf{x}_\perp)\delta(t' - t). \tag{13}$$

This has the consequence that particles behave superdiffusively in the longitudinal direction, with a dynamical exponent $z_\parallel = 4/(5 - d)$ in dimensions $d < d_c = 3$ [20]. In contrast to the previous systems, the superdiffusion here stems from the anomalous, but *equilibrium* wandering of particles in the quenched random velocity field, rather than from interactions that drive the system out of equilibrium.

1.3. Large deviation functions

In each of these systems the goal will be to determine the probability distribution $P(Q,t)$ of the total integrated current in the longitudinal direction $Q(t) = \int_0^t \mathrm{d}t' \int \mathrm{d}^d x\, j_\parallel(\mathbf{x},t')$ up to time t, and the related large deviation function (LDF) $\pi(q)$ defined by

$$\pi(q) = \lim_{t\to\infty} \frac{1}{t} \ln P(Q = qt, t), \tag{14}$$

and expressed in terms of the fluctuating time-averaged current q. For finite-size systems $\pi(q)$ is peaked around its maximum $q_{\mathrm{st}} = \lim_{t\to\infty}\langle Q\rangle/t$, which becomes sharper as the system size increases (i.e., fluctuations decrease with system size). In most physical systems, $\pi(q)$ does not edgewise reduce to a parabolic form—in other words it describes current fluctuations far beyond a Gaussian distribution, even in equilibrium. As such, it appears as a promising tool to probe the distinctive features of NESS as compared to equilibrium states.

For systems driven out of equilibrium by a field E, the fluctuation relation takes the generic form $\pi(-q) = \pi(q) - Eq$. It connects the probabilities of observing longstanding current deviations both close to and far from the steady average q_{st}.

The fluctuations of the longitudinal current are also fully encoded in the moments or the cumulants of Q. We first introduce the generating function

$$Z(s,t) = \langle \mathrm{e}^{-sQ(t)}\rangle = \left\langle \exp\left[-s\int_0^t \mathrm{d}t' \int \mathrm{d}^d x\, j_\parallel(\mathbf{x},t')\right]\right\rangle; \tag{15}$$

moments of Q are then given by the derivatives of $Z(s,t)$ at $s=0$:

$$(-1)^n \frac{\mathrm{d}^n Z(s,t)}{\mathrm{d}s^n}\bigg|_{s=0} = \langle Q(t)^n\rangle. \tag{16}$$

After resorting to the standard Janssen–De Dominicis mapping to a field theory [22–25], we arrive for the systems presented above at the generic form

$$Z(s,t) = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\, \mathrm{e}^{-S[\bar{\phi},\phi,s;t]}. \tag{17}$$

The specific expressions for the actions $S[\bar{\phi},\phi,s;t]$ will be our starting points in the analysis of the systems presented above. We utilize in our work the action functional field theory developed in [22–24], rather than the operator formalism of [30]. These related but distinct methods lead to quite different techniques, as for instance recently exemplified in [31].

A few remarks are in order before we proceed with evaluating the large deviation function for specific systems. Note that the quantity $Z(s,t)$ defined in (15), (17) does play the role of a partition function, whereas the usual dynamic 'partition function' $Z(s=0,t) \equiv 1$ (which expresses the conservation of probability) and carries no relevant information. Moreover, since the action defined in (17) does not describe a Markov process anymore, it is expected that $Z(s,t)$ will grow exponentially with time. It is therefore conventional to introduce a time intensive 'dynamical free energy'

$$\mu(s) = \lim_{t\to\infty} \frac{1}{t} \ln Z(s,t), \tag{18}$$

which is related to the current large deviation function $\pi(q)$ as defined in (14) through a Legendre transform,

$$\pi(q) = \max_s\{\mu(s) + sq\}. \tag{19}$$

Formally, the dynamical free energy $\mu(s)$ is nothing but the generating function for the cumulants of the integrated longitudinal current Q in the asymptotic long-time limit:

$$(-1)^n \frac{\mathrm{d}^n \mu(s)}{\mathrm{d}s^n}\bigg|_{s=0} = \lim_{t\to\infty} \frac{\langle Q(t)^n\rangle_c}{t}. \tag{20}$$

1.4. Steady states with non-zero value of s

In analogy with standard thermodynamics, we observe that the quantity $Z(s,t)$ introduced in (15) facilitates the description of current fluctuations in a *canonical* way (i.e., at fixed intensive parameter s) rather than *microcanonically* (at fixed value of the time-extensive quantity Q).

Besides constituting a useful computational trick, introducing $Z(s,t)$ also provides us with a physical picture of the very configurations that give birth to large deviations. Although at $s \neq 0$ the action defined in (17) does not correspond to a stochastic process anymore, one can understand $S[\bar{\phi},\phi,s;t]$ as describing not a single, but rather an *ensemble* of systems evolving in parallel, with dynamical rules that favour histories with non-zero longstanding deviation of the particle current [32]. The value of physical observables such as correlation functions inferred from $S[\bar{\phi},\phi,s;t]$ at $s \neq 0$ can thus be understood as the typical value of these observables in a *modified* steady state wherein the mean current is enforced to take an average value $q(s)$ that is different from the average $q_{\rm st}$ taken in the steady state. We see for instance from (15) that negative values of s favour histories with an excess current in the direction of the field. Quantitatively, the correspondence between $q(s)$ and s is the same as in the Legendre transform (19). The reader is referred to [32, 33] for further examples and for a numerical exploitation of this picture.

2. Driven diffusive system with mutually exclusive particles and the noisy Burgers equation

2.1. Field-theoretic formulation

We wish to determine the distribution function of the integrated longitudinal current. As outlined in section 1.3, this amounts to determining the 'dynamical free-energy' $Z(s,t)$ defined in (15), rewritten as $Z(s,t) = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,{\rm e}^{-S[\bar{\phi},\phi,s;t]}$. Using periodic boundary conditions we find that the dynamical action $S[\bar{\phi},\phi,s;t]$ is given by

$$S[\bar{\phi},\phi,s;t] = \int {\rm d}^d x \int_0^t {\rm d}t' \Big[\bar{\phi}\big(\partial_{t'} - D\nabla_\perp^2 - D\lambda\nabla_\parallel^2\big)\phi - D(\nabla_\perp\bar{\phi})^2 - D\sigma(\nabla_\parallel\bar{\phi})^2 - \frac{Dsg}{2}\phi^2 + \frac{Dg}{2}(\nabla_\parallel\bar{\phi})\phi^2\Big] + sj_0L^d t - s^2 D\sigma L^d t. \tag{21}$$

(We use the Itô convention as regards time discretization.) It mainly differs from the stochastic action $S[\bar{\phi},\phi,s=0;t]$ through a new quadratic term in the field ϕ, aside from the two deterministic contributions $\sim L^d t$.

In the case of non-interacting particles, each of the cumulants $\langle Q^n\rangle_c$ is extensive in the system volume, which means that the corresponding free energy $\mu_{\rm indep}(s)$ takes the scaling form

$$\mu_{\rm indep}(s,L) = L^d\hat{\mu}_{\rm indep}(s) = L^d(B_1 s + B_2 s^2 + \cdots), \tag{22}$$

where $\hat{\mu}_{\rm indep}(s)$ has a well-defined power expansion around $s=0$. It is known from specific examples, in [7, 35] or out of equilibrium [28, 34], that interactions can lead to an expansion of $\hat{\mu}(s) \equiv L^{-d}\mu(s,L)$ involving non-integer powers of $|s|$ (in the infinite volume limit). Such non-analytic behaviour at $s=0$ simply mirrors the existence of an infinite cumulant of the current as $L\to\infty$. For instance, the LDF for the totally asymmetric exclusion process (TASEP) in the large system limit (with finite density ρ) reads [28, 34]

$$\hat{\mu}_{\rm TASEP}(s) = -\rho(1-\rho)s + \frac{(3\pi)^{\frac{2}{3}}}{5}[\rho(1-\rho)]^{\frac{4}{3}}|s|^{\frac{5}{3}} + \cdots \qquad \text{for} \quad s<0, \tag{23}$$

which means that the variance of the particle current grows faster than the system size, whereas in the symmetric exclusion process (SEP) one has [7, 35]

$$\hat{\mu}_{\mathrm{SEP}}(s) = \frac{1}{2}\rho(1-\rho)s^2 + \frac{2^{\frac{1}{3}}(2\pi)^{\frac{2}{3}}}{5}[\rho(1-\rho)]^{\frac{4}{3}}|s|^{\frac{8}{3}} + \cdots \qquad \text{for} \quad s < 0. \tag{24}$$

The anomalous scaling of the current LDF occurs here for a higher-order cumulant, reflecting that the anomalous distribution of the current arises from sharper details than in the TASEP. From a general point a view, we expect the exponents in (23) or (24) to be universal (for other models within the same universality class), while the prefactors should be model-dependent.

In our case, the continuum and large system size limits will be tackled in section 2.2 with the presentation of the—unavoidably technical—RG analysis of the driven diffusive system we consider. We then proceed to compute the dynamical free energy $\mu(s)$ in section 2.3, before extracting its asymptotic behaviour as a function of s in section 2.4.

2.2. Renormalization

The presence of the additional quadratic vertex in $S[\bar{\phi}, \phi, s; t]$, in spite of allowing for new one-loop diagrams contributing to the three-point vertex function $\Gamma^{(1,2)}$, does not alter the remarkable property of this model that there are strictly no singular loop corrections to $\Gamma^{(1,2)}$, i.e., all new contributions lead to singularities that cancel out precisely. One can explicitly check that order by order in the perturbation expansion, or infer this property from Galilean invariance, see (34) below. The same statements hold for $\Gamma^{(0,2)}$, which receives no singular loop correction either. However, both $\Gamma^{(1,1)}$ and $\Gamma^{(2,0)}$ have vertex corrections given by exactly the same diagrams as with $s = 0$, but with values that are now s-dependent. As a consequence, no new renormalization is required as compared with the stochastic case $s = 0$. Note that for $\sigma = \lambda$ the Einstein relation also holds in the longitudinal sector. Then the symmetry

$$S[\bar{\phi}(t), \phi(t)] = S[\bar{\phi}(-t) - \phi(-t), -\phi(-t)], \tag{25}$$

which for $s = 0$ arises from the existence of a free energy functional wherefrom the Langevin equation derived, continues to hold for $s \neq 0$—in spite of the action not representing a stochastic process anymore—with the consequence that the vertex functions $\Gamma^{(1,2)}$ and $\Gamma^{(0,2)}$ are given by their tree-level expressions.

In the remainder of this subsection, we recall for later use the main lines of the renormalization analysis, which is the same as in the original approach [15, 27]. Denoting by κ an arbitrary momentum scale, the scaling dimensions of the various fields and coupling constants are fixed by the requirement and convenient choice that the action S as well as the diffusivities D and $D\lambda$ be dimensionless,

$$[x] = \kappa^{-1}, \quad [t] = \kappa^{-2}, \quad [q] = \kappa, \quad [\omega] = \kappa^2, \quad [\phi] = [\bar{\phi}] = \kappa^{d/2}, \tag{26}$$

$$[D] = [\sigma] = [\lambda] = \kappa^0, \qquad [g] = \kappa^{1-d/2}, \qquad [s] = \kappa^{1+d/2}. \tag{27}$$

Of course, any other choice for the scaling dimensions of the field $\bar{\phi}$ and the diffusion constant D would eventually lead to the same physical consequences. We infer from the scaling dimension of the nonlinearity the upper critical dimension $d_c = 2$, and also $[sgD] = \kappa^2$. Next we define renormalized fields and parameters via

$$\phi_{\mathrm{R}} = Z_{\phi}^{1/2}\phi, \qquad \bar{\phi}_{\mathrm{R}} = Z_{\bar{\phi}}^{1/2}\bar{\phi}, \tag{28}$$

$$D_{\mathrm{R}} = Z_D D, \qquad \sigma_{\mathrm{R}} = Z_\sigma \sigma, \qquad \lambda_{\mathrm{R}} = Z_\lambda \lambda, \qquad g_{\mathrm{R}} = Z_g g \kappa^{-1+d/2}. \tag{29}$$

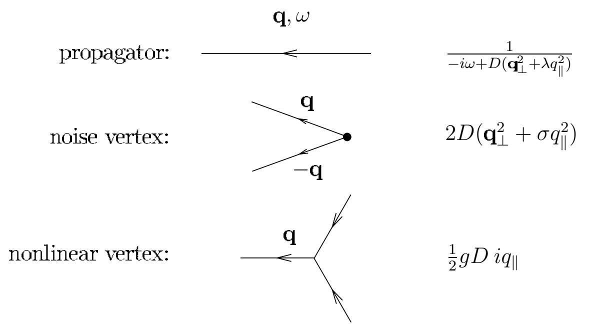

Figure 1. Propagator, noise vertex and nonlinear vertex used in the perturbation expansion for the driven lattice gas.

In the transverse sector, i.e. on the $q_\parallel = 0$ momentum shell, the action is merely Gaussian and we have to all orders in the perturbation expansion

$$\Gamma^{(1,1)}(\mathbf{q}_\perp, q_\parallel = 0, \omega) = \mathrm{i}\omega + D\mathbf{q}_\perp^2, \tag{30}$$

$$\Gamma^{(2,0)}(\mathbf{q}_\perp, q_\parallel = 0, \omega) = -2D\mathbf{q}_\perp^2. \tag{31}$$

As a consequence, $Z_{\bar\phi}^{1/2} Z_\phi^{1/2} = 1$ and $Z_{\bar\phi} = 1$. Then, exploiting that the transverse sector is in equilibrium, we obtain from the associated fluctuation–dissipation theorem the relation $Z_D = Z_{\bar\phi}^{-1/2} Z_\phi^{1/2}$. Thus we see that neither the fields nor D renormalize:

$$Z_{\bar\phi} = Z_\phi = Z_D = 1, \tag{32}$$

and consequently infer the *exact* transverse Gaussian scaling exponents

$$\eta = 0, \qquad z_\perp = 2. \tag{33}$$

The invariance of the action under Galilean transformations

$$\phi'(\mathbf{x}'_\perp, x'_\parallel, t') = \phi(\mathbf{x}_\perp, x_\parallel - Dgu\, t, t) - u \tag{34}$$

shows that ϕ and u must transform in the same way under renormalization (u is the constant velocity of the Galilean transformation). Moreover, the product Dgu also remains invariant under RG. Thus,

$$Z_g = Z_D^{-1} Z_\phi^{-1/2} = 1, \tag{35}$$

and the coupling constant g is not renormalized either.

We now have to perform an explicit loop expansion using the propagator and vertices depicted in figure 1. We use the dimensional regularization scheme and perform the expansion in powers of $\varepsilon = d_c - d = 2 - d$. A straightforward explicit computation of the two-point vertex functions yields

$$Z_\lambda = 1 + \frac{v_\mathrm{R}}{16\varepsilon}(3 + w_\mathrm{R}) + \mathcal{O}\big(v_\mathrm{R}^2\big), \tag{36}$$

$$Z_\sigma = 1 + \frac{v_\mathrm{R}}{32\varepsilon}\big(3w_\mathrm{R}^{-1} + 2 + 3w_\mathrm{R}\big) + \mathcal{O}\big(v_\mathrm{R}^2\big), \tag{37}$$

where we have defined renormalized effective couplings

$$w_\mathrm{R} = \frac{\sigma_\mathrm{R}}{\lambda_\mathrm{R}}, \qquad v_\mathrm{R} = Z_\lambda^{3/2} \frac{g^2}{\lambda^{3/2}} C_d \kappa^{d-2}, \qquad \text{with} \qquad C_d = \frac{\Gamma(2 - d/2)}{2^{d-1}\pi^{d/2}}, \tag{38}$$

and Wilson's flow functions

$$\gamma_\lambda = \kappa\partial_\kappa|_0 \ln\frac{\lambda_R}{\lambda} = -\frac{v_R}{16}(3+w_R), \tag{39}$$

$$\gamma_\sigma = \kappa\partial_\kappa|_0 \ln\frac{\sigma_R}{\sigma} = -\frac{v_R}{32}\left(3w_R^{-1}+2+3w_R\right) \tag{40}$$

to first order in v_R (the subscript '0' here indicates that the derivatives are to be performed in the unrenormalized theory). The RG beta functions follow

$$\beta_w = \kappa\partial_\kappa|_0 w_R = w_R(\gamma_\sigma - \gamma_\lambda) = -\frac{v_R}{32}(w_R-1)(w_R-3), \tag{41}$$

$$\beta_v = \kappa\partial_\kappa|_0 v_R = v_R\left(d-2-\frac{3}{2}\gamma_\lambda\right). \tag{42}$$

RG fixed points are now determined by the zeros of these beta functions. At any non-trivial fixed point $0 < v^\star < \infty$, $w^\star = 1$ is stable and the system reaches effective equilibrium ($\sigma^\star = \lambda^\star$). To *all orders* in perturbation theory we then obtain from (42)

$$\gamma_\lambda^\star = \gamma_\sigma^\star = \frac{2}{3}(d-2), \tag{43}$$

in dimensions $d \leqslant d_c = 2$.

The dynamical correlation function scales as

$$C(\mathbf{q},\omega) = \mathbf{q}_\perp^{-z_\perp-2+\eta}\hat{C}\left(\frac{q_\parallel}{|\mathbf{q}_\perp|^{1+\Delta}}, \frac{\omega}{|\mathbf{q}_\perp|^z}\right), \tag{44}$$

where in $d \leqslant 2$ dimensions *exactly*

$$\Delta = -\frac{\gamma_\lambda^\star}{2} = \frac{2-d}{3}. \tag{45}$$

We thus obtain

$$z_\parallel = \frac{z_\perp}{1+\Delta} = \frac{6}{5-d}, \tag{46}$$

and as expected, $z_{(\parallel)} = 3/2$ in one dimension. For $d > 2$ the system scales towards the Gaussian fixed point $v_0^* = 0$, whence one arrives at the mean-field scaling exponents $\Delta = 0$, $z_\parallel = z_\perp = 2$.

2.3. *Evaluation of the cumulant generating function* $\mu(s)$

Having recalled the main results of the RG analysis, we can now turn to the evaluation of the dynamical free energy $\mu(s)$. The dynamical action (21) contains an extensive (in time as well) deterministic contribution, $L^d t(sj_0 - s^2 D\sigma)$, and a fluctuating one, for which it will prove sufficient to extract the tree level approximation. Quite unfamiliarly in dynamical field theory, the latter contribution arises from the normalization of the path integral. The Gaussian contribution to the action $S[\bar\phi, \phi, s; t]$ takes the form $\int \frac{1}{2}(\bar\phi\phi)\Gamma_0\binom{\bar\phi}{\phi}$ with, conveniently written in Fourier space,

$$\Gamma_0(\mathbf{q},\omega) = \begin{pmatrix} -2D\left(\mathbf{q}_\perp^2+\sigma q_\parallel^2\right) & -\mathrm{i}\omega + D\left(\mathbf{q}_\perp^2+\lambda q_\parallel^2\right) \\ \mathrm{i}\omega + D\left(\mathbf{q}_\perp^2+\lambda q_\parallel^2\right) & -Dgs \end{pmatrix}. \tag{47}$$

We have

$$\det\Gamma_0 = -\left[\omega^2 + \Omega_s^2(\mathbf{q})\right], \tag{48}$$

$$\Omega_s^2(\mathbf{q}) = D^2\big(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\big)^2 - 2sgD^2\big(\mathbf{q}_\perp^2 + \sigma q_\parallel^2\big), \tag{49}$$

and integrating over the fields yields the following tree level contribution to $\mu(s)$:

$$\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\left[-\int \frac{1}{2}(\bar{\phi}\phi)\Gamma_0\begin{matrix}\bar{\phi}\\ \phi\end{matrix}\right] \tag{50}$$

$$= \exp\left[-\frac{1}{2}L^d t\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}\omega}{2\pi}\ln\left|\det\frac{\Gamma_0(\mathbf{q},\omega)}{\Gamma_0(\mathbf{q},\omega)|_{s=0}}\right|\right] \tag{51}$$

$$= \exp\left[-\frac{1}{2}L^d t\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}\omega}{2\pi}\ln\left(1 - \frac{2sgD^2\big(\mathbf{q}_\perp^2 + \sigma q_\parallel^2\big)}{\omega^2 + D^2\big(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\big)^2}\right)\right] \tag{52}$$

$$= \exp\left(-\frac{1}{2}L^d tD\int \frac{\mathrm{d}^d q}{(2\pi)^d}\left[\sqrt{\big(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\big)^2 - 2sg\big(\mathbf{q}_\perp^2 + \sigma q_\parallel^2\big)} - \big(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\big)\right]\right). \tag{53}$$

At the RG fixed point the values of λ and σ are not fixed, but become equal ($w^\star = 1$). Without loss of generality, one can thus start from equal bare coupling constants $\sigma = \lambda$ (as was for instance implicitly assumed in [29]). The last integral (53) can now be carried out explicitly, and its infrared divergence is isolated by extracting the first term of the small s expansion. This gives

$$L^{-d}\mu(s) = -j_0 s + s^2 D\sigma + \frac{1}{2}sgD\Lambda^d + \frac{D\lambda^{-1/2}(-gs)^{1+d/2}\Gamma(d)\Gamma(-1-d/2)}{2^d(2\pi)^{d/2}[\Gamma(d/2)]^2}. \tag{54}$$

2.4. Scaling behaviour of $\mu(s)$

To determine the expansion of $\mu(s)$ in powers of s in the continuum limit, and at the stable RG fixed point $w^\star = 1$ ($\lambda = \sigma$) we note that making explicit the $1/\varepsilon$ pole in (54),

$$L^{-d}\mu(s) = -j_0 s + s^2 D\lambda\left[1 + \frac{C_d g^2}{\lambda^{3/2}}\frac{(-gs)^{-\varepsilon/2}}{4\varepsilon}\right], \tag{55}$$

one recognizes the expression for $Z_\lambda = 1 + \frac{v_R}{4\varepsilon}$ and finds (at the convenient normalization point $\kappa = (-sg)^{\frac{1}{2}}$)

$$L^{-d}\mu(s) = -j_0 s + D\lambda_R s^2. \tag{56}$$

Finally, using that in the vicinity of the RG fixed point $\lambda_R \sim \lambda\kappa^{\gamma_\lambda^\star}$, with the anomalous scaling dimension $\gamma_\lambda^\star = \gamma_\sigma^\star = -\frac{2}{3}\varepsilon$ of the noise correlator, the asymptotic scaling of $\mu(s)$ as a function of s reads (below the critical dimension $d_c = 2$)

$$L^{-d}\mu(s) = -j_0 s + \mathcal{A}_d s^{2-\varepsilon/3} = -j_0 s + \mathcal{A}_d s^{(d+4)/3}. \tag{57}$$

The constant prefactor $\mathcal{A}_d$ is non-universal, contrary to the exponent $2-\varepsilon/3 = (d+4)/3$, which holds exactly to all orders in perturbation theory. In one dimension, we recover the exponent 5/3 of the totally asymmetric exclusion process [28, 34], see (23). This result expectedly corroborates that the TASEP and systems described by the Burgers equation belong to the same (dynamical) universality class.

We emphasize that the scaling exponent in (57) merely follows from the noise renormalization and could thus have been obtained through straightforward dynamic scaling. A simpler, though less explicit, way to arrive at that conclusion is to employ the matching condition stemming from the Λ^d IR divergence in (54). The only factor acquiring an anomalous dimension in the crucial second term in (54) is the parameter σ; with $\gamma_\sigma^\star = \frac{2}{3}(d-2)$ and the matching condition $\kappa \sim (-sg)^{1/2}$ that directly follows from the scaling dimensions, we again arrive at $L^{-d}\mu(s) + j_0 s \sim s^{2+(d-2)/3}$. Indeed, this will be our method of choice in the following subsection as well as for the remaining two systems.

2.5. Logarithmic corrections to $\mu(s)$ in two dimensions

Precisely at the upper critical dimension $d_c = 2$, the RG flow equations provide access to the logarithmic corrections to the mean-field critical power laws. In our context, they also specify the logarithmic correction to the s^2 term in $\mu(s)$. To determine how the renormalized coupling constants vary under scale transformation $\kappa \mapsto \ell\kappa$, let us recall that running couplings $\tilde{\lambda}(\ell)$ and $\tilde{v}(\ell)$ are defined through

$$\ell \frac{\mathrm{d}\tilde{\lambda}(\ell)}{\mathrm{d}\ell} = \gamma_\lambda(\ell)\tilde{\lambda}(\ell), \qquad \ell \frac{\mathrm{d}\tilde{v}(\ell)}{\mathrm{d}\ell} = \beta_v(\ell), \tag{58}$$

with initial conditions $\tilde{\lambda}(1) = \lambda_{\mathrm{R}}$, $\tilde{v}(1) = v_{\mathrm{R}}$, and where $\gamma_\lambda(\ell) = \gamma_\lambda(\tilde{v}(\ell))$, $\beta_v(\ell) = \beta_v(\tilde{v}(\ell))$. In $d_c = 2$ dimensions, the flow equation for $\tilde{v}(\ell)$ reads

$$\ell \frac{\mathrm{d}\tilde{v}(\ell)}{\mathrm{d}\ell} = \frac{3}{8}\tilde{v}^2(\ell) + \mathcal{O}(\tilde{v}^3(\ell)), \tag{59}$$

which is solved by

$$\tilde{v}(\ell) = \frac{v_{\mathrm{R}}}{1 - \frac{3}{8} v_{\mathrm{R}} \ln \ell}. \tag{60}$$

Upon inserting this result in the flow equation for $\tilde{\lambda}(\ell)$, one obtains

$$\tilde{\lambda}(\ell) \sim \bar{\lambda}(\ln \ell)^{2/3}, \tag{61}$$

which gives the precise form of (57) at $d_c = 2$:

$$L^{-2}\mu(s) = -j_0 s + \mathcal{A}_2 s^2(-\ln|s|)^{2/3}. \tag{62}$$

Of course, such logarithmic corrections are familiar within the framework of RG calculations at the upper critical dimension. Yet we stress that this result constitutes the first *exact* result for the LDF scaling of a two-dimensional system, where the methods of integrable systems fail to apply.

2.6. Correlation functions

In this subsection, we aim at determining how the correlation function is modified for small (negative) values of s. As announced in section 1.4, states with $s < 0$ are characteristic of histories with excess current in the direction of the driving field. Let $G^{(m,n)} = \langle \bar{\phi}^m \phi^n \rangle$ denote the m-$\bar{\phi}$ and n-ϕ correlation function. For the complete quadratic part of the action, we find that

$$\begin{aligned} G^{(1,1)}(\mathbf{q}, t, t') &= \overbrace{\Theta(t'-t)\frac{\Omega_0 + \Omega_s}{2\Omega_s}\mathrm{e}^{-\Omega_s(t'-t)}}^{\text{causal part}} + \overbrace{\Theta(t-t')\frac{\Omega_0 - \Omega_s}{2\Omega_s}\mathrm{e}^{-\Omega_s(t-t')}}^{\text{non-causal part}}, \\ G^{(0,2)}(\mathbf{q}, t, t') &= \frac{D\big(\mathbf{q}_\perp^2 + \sigma q_\parallel^2\big)}{\Omega_s}\mathrm{e}^{-\Omega_s|t'-t|}, \\ G^{(2,0)}(\mathbf{q}, t, t') &= \frac{Dsg}{2\Omega_s}\mathrm{e}^{-\Omega_s|t'-t|}, \end{aligned} \tag{63}$$

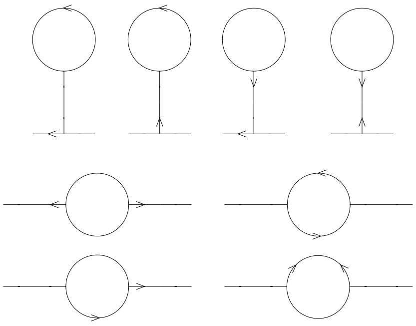

Figure 2. One-loop graphs which renormalize the correlation function in the s-state.

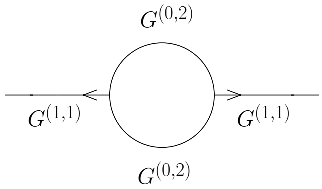

Figure 3. Single diagram contributing to $C(\mathbf{q}, t, t)$ in the s-state, at minimal order with respect to the external momentum $\mathbf{q}$. The expressions for the propagators are given in (63).

with the abbreviation (49), and with the convention that the step function satisfies $\Theta(0) = 0$. (Note that the above calculations hold only for $s < 0$.) The diagrammatic expansion contains no causality constraint on the loops anymore. The long-range spatial correlations in the s-state are obtained from an expansion in $\mathbf{q}$ at minimal order. To one loop, the correlation function is renormalized through the graphs shown in figure 2. To lowest order in $\mathbf{q}$, we find that all contributions cancel except for the single loop diagram depicted in figure 3, which yields

$$C(\mathbf{q}, t, t) - C(\mathbf{q}, t, t)|_{s=0} = (-2gs)^{-1/2}\left(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\right)^{1/2} \times \left[1 + g^2\lambda^{-1/2}(-gs)^{-\varepsilon/2}\frac{1}{16\pi\varepsilon}\frac{q_\parallel^2}{\mathbf{q}_\perp^2 + \lambda q_\parallel^2}\right]. \qquad (64)$$

Averaging along the equilibrium directions, i.e., at $q_\perp = 0$, one recognizes

$$C(q_\parallel, 0, t, t) - C(q_\parallel, 0, t, t)|_{s=0} = (-2gs)^{-1/2}\lambda_{\mathrm{R}}^{1/2} q_\parallel, \qquad (65)$$

and finally obtains the exponent of s in the s-correlation function:

$$C(q_\parallel, 0, t, t) - C(q_\parallel, 0, t, t)|_{s=0} \propto (-s)^{-1/2-\varepsilon/6}. \qquad (66)$$

In one dimension, this gives the power law $(-s)^{-2/3}$, which, to our knowledge, has not been obtained through exact approaches for systems in the Burgers/KPZ universality class.

3. A driven diffusive system with a continuous phase transition

3.1. Renormalization

The previous considerations and calculations can readily be generalized to the critical driven diffusive system described by the nonlinear Langevin equation (8) with noise correlator (6).

Since only the transverse sector in momentum space is rendered critical, we merely need to replace the propagator in figure 1 with the expression $1/[-\mathrm{i}\omega + D\mathbf{q}_\perp^2(r + \mathbf{q}_\perp^2) + D\lambda q_\parallel^2]$. Dimensional analysis yields $[r] = \kappa^2$, and at criticality $(r = 0)$ $[q_\parallel] = [q_\perp]^2 = \kappa^2$, and hence $[\omega] = \kappa^4$. This yields for the scaling dimension of the nonlinear coupling $[g] = \kappa^{(5-d)/2}$, wherefrom we infer the upper critical dimension $d_c = 5$.

Renormalizing this model proceeds in much the same way as described above in section 2.2 (see [18, 27]). Since again the nonlinear vertex is proportional to $\mathrm{i}q_\parallel$, the transverse sector is Gaussian with

$$\Gamma^{(1,1)}(\mathbf{q}_\perp, q_\parallel = 0, \omega) = \mathrm{i}\omega + D\mathbf{q}_\perp^2(r + \mathbf{q}_\perp^2), \tag{67}$$

$$\Gamma^{(2,0)}(\mathbf{q}_\perp, q_\parallel = 0, \omega) = -2D\mathbf{q}_\perp^2. \tag{68}$$

Consequently (32) holds here as well, implying that

$$\eta = 0, \qquad \nu = \tfrac{1}{2}, \qquad z_\perp = 4 \tag{69}$$

exactly. The system is of course still invariant with respect to Galilean transformations, whence (35) is valid. The RG beta function for the effective nonlinearity $v = g^2/\lambda^{3/2}$ now reads

$$\beta_v = v_\mathrm{R}\left(d - 5 - \frac{3}{2}\gamma_\lambda\right), \tag{70}$$

and thus again to *all* orders in the perturbation expansion

$$\gamma_\lambda^\star = \tfrac{2}{3}(d - 5), \tag{71}$$

and consequently

$$\Delta = 1 - \frac{\gamma_\lambda^\star}{2} = \frac{8 - d}{3}, \qquad z_\parallel = \frac{z_\perp}{1 + \Delta} = \frac{12}{11 - d}. \tag{72}$$

3.2. *Evaluation and scaling behaviour of $\mu(s)$*

The Gaussian contribution to the action $S[\bar{\phi}, \phi, s; t]$ now becomes

$$\exp\left[-\frac{1}{2}L^d t \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}\omega}{2\pi} \ln\left(1 - \frac{2sgD^2(\mathbf{q}_\perp^2 + \sigma q_\parallel^2)}{\omega^2 + D^2[\mathbf{q}_\perp^2(r + \mathbf{q}_\perp^2) + \lambda q_\parallel^2]^2}\right)\right]. \tag{73}$$

Thus we observe that we require the scaling dimension of the noise strength σ, which is an *irrelevant* parameter, in the RG sense, for this model. A straightforward one-loop calculation gives

$$\gamma_\lambda = -v_\mathrm{R}, \qquad \gamma_\sigma = 2 - \frac{v_\mathrm{R}}{3}, \tag{74}$$

so for $\epsilon = 5 - d > 0$,

$$\gamma_\sigma^\star = 2 - \frac{2\epsilon}{9} + \mathcal{O}(\epsilon^2). \tag{75}$$

However time now scales according to $[t] = \kappa^{-4}$, and the appropriate matching condition becomes $\kappa \sim (-sg)^{1/4}$. Thus we finally arrive at

$$L^{-d}\mu(s) = -j_0 s + \mathcal{B}_d s^{2-\epsilon/18}, \tag{76}$$

with a non-universal amplitude $\mathcal{B}_d$. Note that this result holds only to first order in the dimensional expansion for $d = 5 - \epsilon < 5$.

At the critical dimension $d_c = 5$,

$$\tilde{v}(\ell) = \frac{v_R}{1 - \frac{3}{2} v_R \ln \ell}, \qquad \tilde{\sigma}(\ell) \sim \bar{\sigma}\,\ell^2 (\ln \ell)^{2/9}, \tag{77}$$

wherefrom we obtain the logarithmic correction

$$L^{-5}\mu(s) = -j_0 s + \mathcal{B}_5 s^2 (-\ln|s|)^{2/9}. \tag{78}$$

4. Superdiffusion in a random velocity field

4.1. Mesoscopic formulation

Before embarking on the determination of $\mu(s)$ for the case of (super-)diffusive particles subject to a random velocity field, we have to establish how the random field ψ intervenes in the definition of the LDF of interest. The physical features of the current fluctuations are contained in the disorder-averaged *cumulants* $\overline{\langle Q(t)^n \rangle_c}$. Here and below the averaging over thermal disorder will be noted by $\langle \cdots \rangle$, while the average over the random field will be indicated with an overbar, $\overline{\ldots}$. We are thus ultimately interested in determining the LDF

$$\mu(s) = \lim_{t\to\infty} \frac{1}{t} \overline{\ln Z_\psi(s,t)}. \tag{79}$$

The disorder-dependent partition function $Z_\psi(s,t)$ reads

$$Z_\psi(s,t) = \left\langle \exp\left[-s \int_0^t \mathrm{d}t'\, \mathrm{d}^d x\, j_\parallel(\mathbf{x}, t') \right] \right\rangle, \tag{80}$$

and as in the previous models, a field-theoretic reformulation of (9)–(11) and (80) enables us to rewrite the partition function as

$$Z_\psi(s,t) = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, \mathrm{e}^{-S_\psi[\bar{\phi},\phi,s;t]}, \tag{81}$$

where the quenched and s-dependent action is given by

$$S_\psi[\bar{\phi},\phi,s;t] = -s^2 D_\parallel L^d t + \int \mathrm{d}^d x \int_0^t \mathrm{d}t' \big[\bar{\phi}\big(\partial_{t'} - D_\perp \nabla_\perp^2 - D_\parallel \nabla_\parallel^2 - \psi \nabla_\parallel\big)\phi \\ - D_\perp (\nabla_\perp \bar{\phi})^2 - D_\parallel (\nabla_\parallel \bar{\phi})^2 - s\psi\phi \big]. \tag{82}$$

The form of this action is related to Honkonen's original ($s = 0$) action [20] through the canonical transformation

$$\bar{\phi} = \bar{\rho}, \qquad \phi = \rho + \bar{\rho}, \tag{83}$$

whereupon S_ψ takes the form

$$S_\psi[\bar{\rho},\rho,s;t] = -s^2 D_\parallel L^d t \\ + \int \mathrm{d}^d x \int_0^t \mathrm{d}t' \big[\bar{\rho}\big(\partial_{t'} - D_\perp \nabla_\perp^2 - D_\parallel \nabla_\parallel^2 - \psi \nabla_\parallel\big)\rho + s(\bar{\rho} + \rho)\psi \big]. \tag{84}$$

For $s = 0$ this is precisely the action that was studied by Honkonen [20].

In a microscopic lattice gas formulation (independent particles diffusing in a random field), $Z_\psi(s,t)$ can also be obtained from the standard Doi–Peliti approach [36]. Up to irrelevant terms, one arrives again at the action (82), with the field ϕ being related to the original operators through a Cole–Hopf transformation. This illustrates that, either in the Janssen–De Dominicis or in the Doi–Peliti manner, the well-tried functional methods of statistical physics can be readily extended to tackle large deviation functions and the associated s-modified states, as previously proposed in [37].

4.2. Renormalization

The theory has two superficially divergent graphs which lead to the same renormalization of $D_\parallel$. As can be seen from direct inspection of possible diagrams, the additional $s\psi\phi$ term in the action does not lead to further renormalization, as compared to the $s = 0$ case. We define as in [20] renormalized couplings through $D_\parallel = ZD_{\parallel,\mathrm{R}}$, $\lambda = \lambda_\mathrm{R}\kappa^\epsilon$. The perturbative expansion is organized in powers of a single dimensionless parameter u_R that we fix to

$$u_\mathrm{R} = \frac{\lambda_\mathrm{R}}{2\pi D_{\parallel,\mathrm{R}} D_\perp}. \tag{85}$$

Examination of the one-loop contribution to $\Gamma^{(1,1)}$ leads to

$$Z = 1 - \frac{u_\mathrm{R}}{\epsilon}, \qquad \beta_u = u_\mathrm{R}(u_\mathrm{R} - \epsilon). \tag{86}$$

Below the critical dimension $d_c = 3$, the RG flow has a single non-trivial fixed point $u_\mathrm{R}^\star = \epsilon$, at which the system exhibits superdiffusion [20], namely

$$\left\langle x_\parallel^2 \right\rangle \sim t^{1+\epsilon/2}; \tag{87}$$

in other words, the dynamical exponent in the longitudinal direction is

$$z_\parallel = \frac{4}{2+\epsilon}. \tag{88}$$

4.3. Determination of $\mu(s)$

From a dynamical point of view, the study of disordered systems does not pose any particular problem since conservation of probability ensures that the dynamical partition function remains equal to 1 as time evolves. This has the well-known consequence that quenched disorder can be treated without resorting to the replica trick. However, in the present case, our interest aims precisely at the nonzero and s-dependent dynamical partition function $Z_\psi(s, t)$, which grows exponentially with time, with a disorder-dependent rate. The dynamical free energy we pursue, denoted by $\mu(s)$, is obtained as the average of this rate, see (79). The replica trick represents one convenient means to achieve this disorder averaging and to determine $\mu(s)$ via

$$\mu(s) = \lim_{t\to\infty} \frac{1}{t}\frac{\partial}{\partial n}\bigg|_{n=0} \overline{[Z_\psi(s,t)]^n}. \tag{89}$$

In contrast with previous work [38] using replicas to probe dynamical aspects of disordered systems, our approach leads to non-trivial couplings between replicas. Integration over ψ yields

$$\overline{[Z_\psi(s,t)]^n} = \int \mathcal{D}\bar\phi_a \mathcal{D}\phi_a\, \mathrm{e}^{-S_n[\bar\phi,\phi,s;t]}, \tag{90}$$

with an effective action $S_n[\bar\phi_a, \phi_a, s; t]$ whose quadratic part reads

$$S_n[\bar\phi_a, \phi_a, s; t] = -nL^d t D_\parallel s^2 - s^2\frac{\lambda}{2}\sum_{a,b}\int \frac{\mathrm{d}^{d-1}\mathbf{q}_\perp}{(2\pi)^{d-1}}\phi_a(\mathbf{q}_\perp,0,0)\phi_b(-\mathbf{q}_\perp,0,0)$$
$$+\sum_a \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{\mathrm{d}\omega}{(2\pi)}\left[\bar\phi_a(-\mathrm{i}\omega + D\cdot\mathbf{q}^2)\phi_a - (D\cdot\mathbf{q}^2)\bar\phi_a^2\right], \tag{91}$$

where we have set $D\cdot\mathbf{q}^2 = D_\perp(\mathbf{q}_\perp)^2 + D_\parallel(q_\parallel)^2$ for clarity. Integration over the noise has given birth to a uniform coupling between replicas of the field ϕ, in the $\omega = 0$, $q_\perp = 0$ sector. However, no new renormalization is needed, as can be directly checked.

The free energy $\mu(s)$ is obtained as the ratio of the determinant of the action at $s \neq 0$ and $s = 0$. In our case, the effective action (91) is diagonal in Fourier space, and the only modes which depend on s are those with $q_\parallel = 0$ and $\omega = 0$. We thus require the matrix determinant

$$\Delta_n(\mathbf{q}_\perp) = \det \begin{pmatrix} -2D_\perp \mathbf{q}_\perp^2 \mathbb{I}_n & D_\perp(\mathbf{q}_\perp^2 + m^2)\mathbb{I}_n \\ D_\perp(\mathbf{q}_\perp^2 + m^2)\mathbb{I}_n & -\lambda s^2 \mathbb{J}_n \end{pmatrix}, \tag{92}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix, and $\mathbb{J}_n$ represents the matrix whose entries are all 1. For later convenience, we have also added a mass m to the $\bar{\phi}\phi$ term. Using standard results about block determinants, we find

$$\Delta_n(\mathbf{q}_\perp) = (-1)^n \left(D_\perp \mathbf{q}_\perp^2\right)^{2n} \left[D_\perp^2\left(\mathbf{q}_\perp^2 + m^2\right)^2 - 2n\lambda s^2 D_\perp \mathbf{q}_\perp^2\right]. \tag{93}$$

The corresponding (tree-level) contribution to $\mu(s)$ is then given by

$$-\frac{1}{2}L^d \frac{\partial}{\partial n}\bigg|_0 \int \frac{\mathrm{d}^{d-1}\mathbf{q}_\perp}{(2\pi)^{d-1}} \ln \frac{\Delta_n(\mathbf{q}_\perp)}{\Delta_n(\mathbf{q}_\perp)|_{s=0}} = L^d \frac{\lambda s^2}{D_\perp} \int \frac{\mathrm{d}^{d-1}\mathbf{q}_\perp}{(2\pi)^{d-1}} \frac{q_\perp^2}{\left(q_\perp^2 + m^2\right)^2}$$
$$= L^d \frac{\lambda s^2 (2-\epsilon) m^{-\epsilon}}{4(2\pi)^{2-\epsilon}} \Gamma\left(\frac{\epsilon}{2}\right) \Gamma\left(1 - \frac{\epsilon}{2}\right). \tag{94}$$

Combining this result with the deterministic contribution to $\mu(s)$ we recognize

$$\mu(s) = L^d D_\parallel s^2 \left(1 + \frac{u_\mathrm{R}}{\epsilon}\right) = L^d D_{\parallel,\mathrm{R}} s^2. \tag{95}$$

4.4. Scaling behaviour of $\mu(s)$

As in the previous examples, we infer from (95) that the continuum limit in $\mu(s)$ is determined by the renormalization of the noise coupling constant. Using $\gamma^\star_{D_\parallel} = -\epsilon$, one gets

$$L^{-d}\mu(s) \sim s^2 \kappa^{-\epsilon}, \tag{96}$$

but from the explicit expression (92), the normalization point depends on s only through λs^2. Since $[\lambda s^2] = \kappa^5$, we obtain the scaling behaviour, valid below the critical dimension $d_c = 3$,

$$L^{-d}\mu(s) \sim s^{2-2\epsilon/5}. \tag{97}$$

We note that although this system is in equilibrium, the expansion (97) implies that that in a finite-size system the variance of the current grows with the linear size L as L^3 for $d < 3$, in contrast to the L^d behaviour observed in normally diffusive systems. Again, this anomalous behaviour is exemplified here for the first time.

At the critical dimension $d_c = 3$, the running coupling

$$\tilde{D}_\parallel(\ell) \sim D_\mathrm{R} \ln \ell, \tag{98}$$

which gives the logarithmic correction

$$\mu(s) \sim s^2(-\ln|s|). \tag{99}$$

At the upper critical dimension too, the RG analysis leads to an asymptotically exact result.

4.5. Correlation function

As well as allowing access to free energies, the replica trick can also be used to determine the correlation functions, averaged over thermal fluctuations and the random field. This is achieved by noting that

$$\overline{\langle \phi(\mathbf{q}_\perp, 0, 0)\phi(-\mathbf{q}_\perp, 0, 0)\rangle_c} = \overline{\langle \phi_a(\mathbf{q}_\perp, 0, 0)\phi_a(-\mathbf{q}_\perp, 0, 0)\rangle_c}|_{n\to 0}. \tag{100}$$

Thus we obtain at the tree level the correlation function in the s-state as

$$\langle\phi(\mathbf{q}_\perp,0,0)\phi(-\mathbf{q}_\perp,0,0)\rangle_c = \frac{2}{D_\perp \mathbf{q}_\perp^2}\left(1-\lambda s^2\frac{2}{D_\perp \mathbf{q}_\perp^2}\right). \tag{101}$$

The spatio-temporal correlation function reads

$$C(\mathbf{x},t) = C(\mathbf{x},t)|_{s=0} - \frac{2\lambda s^2}{D_\perp}\int \frac{d^{d-1}}{(2\pi)^{d-1}}\frac{1}{\mathbf{q}_\perp^2}\mathrm{e}^{\mathrm{i}\mathbf{x}_\perp\mathbf{q}_\perp}, \tag{102}$$

i.e., the supplementary correlations in the s-steady state are still uniform in time and along the longitudinal direction, which means that states conveying a (slightly) atypical value of the current do not break these symmetries of the steady state. Including loop corrections to the correlation function does not alter this result, but affects the power s^2 in (102), as a consequence of superdiffusivity.

5. Conclusions

In the previous sections, we have shown that the implementation of dynamical RG techniques to the already well-studied (see [2–14]) dynamical free energy $\mu(s)$ associated with the particle current provides new insights into the singular dynamical *macroscopic limit* of large systems. As is the case for the static free energies in thermal equilibrium setups, $\mu(s)$ exhibits singularities only in the infinite volume limit. In the systems presented in this work, $\mu(s)$ becomes non-analytic at $s = 0$, where s is the parameter canonically conjugated to the particle current:

$$L^{-d}\mu(s) + j_0 s \sim s^a, \qquad \text{with} \qquad 1 < a \leqslant 2. \tag{103}$$

In finite-size systems, the variance of the integrated current grows with the system size faster than linearly. We have shown for the systems under consideration here that the exponents a were simply related to the respective dynamical exponents z through the anomalous exponent for the noise strengths. In our examples, the divergence of the variance ($a < 2$) emerges as a direct consequence of superdiffusion ($z < 2$).

Our second system provides an example for a current LDF in presence of twofold, namely static *and* dynamic criticality. Our final model system constitutes the first example of an *equilibrium* system for which the fluctuations of the particle current were shown to produce anomalous scaling with system size.

In previous works, which remained confined to a specific one-dimensional system, namely the asymmetric exclusion process, an exponent related to a was determined using the Bethe ansatz [28, 33, 34]. Of course, some of these existing results can be recovered through our field-theoretic methods employed here (such as the scaling exponent in $d = 1$), but we have obtained additional asymptotically exact results in higher dimensions. As well as opening the path to a wide range of applications, our method also connects to other multi-purpose instruments of statistical physics (such as replicas and supersymmetry for disordered systems) that can be deftly adapted to describe the s-dependent states. As illustrated in sections 2.6 and 4.5, our method also gives access to the correlation functions in the s-state, which to our knowledge cannot be obtained from Bethe ansatz computations, but still provides valuable physical information about the critical behaviour in the vicinity of $s = 0$.

Finally we remark that for the driven systems investigated here we have not been able to access the $s > 0$ states which probe realizations of the process with a current *opposite* to the average current. This is likely because instead of remaining homogeneous in time and space, the typical trajectories develop heterogeneities in this situation. We may speculate that, as discussed by Bodineau and Derrida [10, 11], non-stationary profiles, perhaps with shock-like spatial structures, will play an important role. This issue would certainly be worth further detailed investigations.

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