Large deviations in and out of equilibrium

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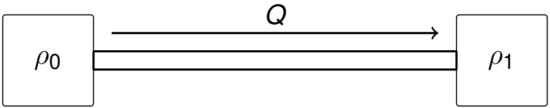
La Herradura - Granada Seminar – 12th September 2010

Motivations



$$\mathrm{Prob}(\mathcal{C}) \propto \exp\left\{-\frac{\mathrm{energy}(\mathcal{C})}{\mathrm{temperature}}\right\}$$ cannot describe

- Non-equilibrium steady-state

$$\mathrm{Prob}[\rho(x)]$$

- Equilibrium fluctuations of dynamical observables

$$\mathrm{Prob}[Q]$$

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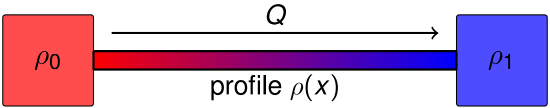

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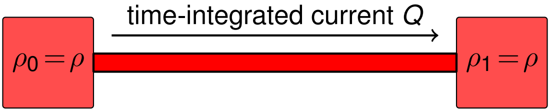

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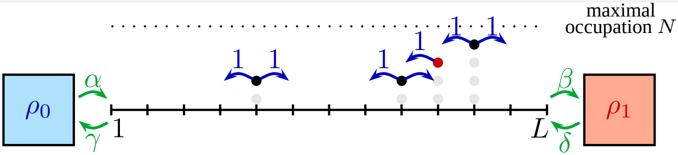
Exclusion processes



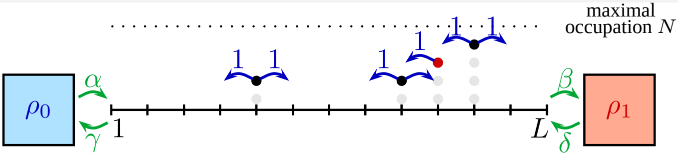
- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution

$$\partial_t P(\{n_i\}) = \sum_{n_i'} \left[W(n_i' \to n_i) P(\{n_i'\}) - W(n_i \to n_i') P(\{n_i\}) \right]$$

- Large deviation function of the time-integrated current Q

$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \qquad (\Leftrightarrow \text{determining } P(Q))$$

Exclusion processes



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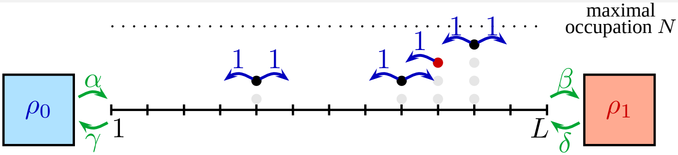

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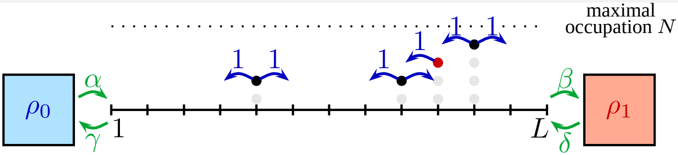
Operator representation

[Schütz & Sandow PRE **49** 2726]



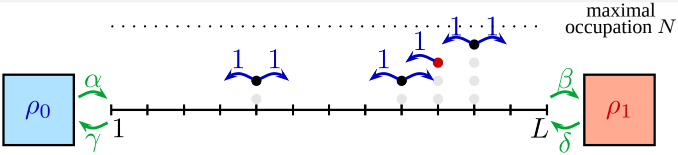
$$\partial_t P \;=\; \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\leq k\leq L-1} \Big[S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k) \Big]$$
$$+\,\alpha\left[S_1^+ - (1-\hat{n}_1)\right] + \gamma\left[S_1^- - \hat{n}_1\right]$$
$$+\,\delta\left[S_L^+ - (1-\hat{n}_L)\right] + \beta\left[S_L^- - \hat{n}_L\right]$$

$S^{\pm}$ and **creation and annihilation operators**:

$$S^+|n\rangle = (N-n)|n+1\rangle \qquad S^-|n\rangle = n|n-1\rangle$$

Operator representation [Schütz & Sandow PRE **49** 2726]



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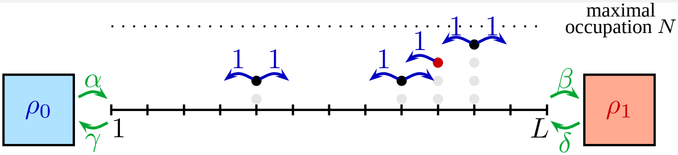
$S^\pm$ and **creation and annihilation operators**:

$$S^+|n\rangle = (N-n)|n+1\rangle \qquad S^-|n\rangle = n|n-1\rangle$$

$S^\pm$ and $S^z = \hat{n} - \frac{N}{2}$ are **spin operators** (with $j = \frac{N}{2}$)

Large deviation function



$$\left\langle e^{-sQ} \right\rangle \sim e^{t\psi(s)} \qquad \text{with} \qquad \psi(s) = \max \operatorname{Sp} \mathbb{W}(s)$$

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Mapping non-eq to eq [Imparato, VL, van Wijland, arXiv:0911.0564]

Large deviations of the current

$$\psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

$$\mathbb{W}(s) = \overbrace{\sum_{1 \le k \le L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}}$$
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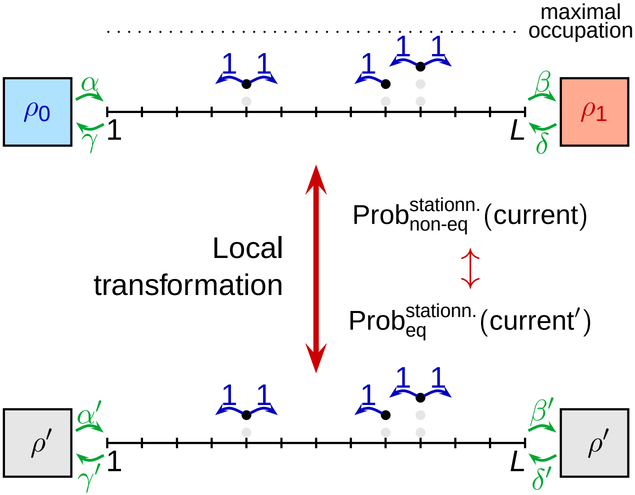
Local transformation

$$\mathcal{Q}^{-1}\mathbb{W}(s)\mathcal{Q} = \sum_{1\le k\le L-1} \vec{S}_k \cdot \vec{S}_{k+1}$$
$$+ \alpha' \left[S_1^+ - (1-\hat{n}_1)\right] + \gamma' \left[S_1^- - \hat{n}_1\right]$$
$$\underbrace{+\delta' \left[S_L^+ e^{s'} - (1-\hat{n}_L)\right] + \beta' \left[S_L^- e^{-s'} - \hat{n}_L\right]}_{\text{describes contact with reservoirs of same densities}}$$

For the current

[Imparato, VL, van Wijland, **PRE** 80 011131]

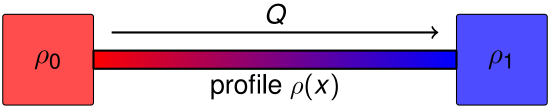
Symmetric exclusion process



System in equilibrium

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]



A reminder: propagator in quantum mechanics

$\langle \text{final}| e^{it\mathbb{H}} |\text{initial}\rangle$

Macroscopic limit

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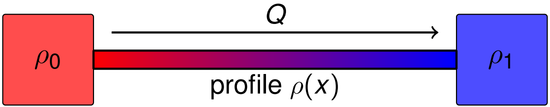

A reminder: propagator in quantum mechanics

$$\langle \text{final}| e^{it\mathbb{H}} |\text{initial}\rangle = \int dz_1 \ldots dz_n \langle \text{final}| e^{i\Delta t\mathbb{H}} |\underline{z_n}\rangle \langle \underline{z_{n-1}}| e^{i\Delta t\mathbb{H}} |\underline{z_{n-2}}\rangle \ldots$$
$$\ldots \langle \underline{z_1}| e^{i\Delta t\mathbb{H}} |\text{initial}\rangle$$

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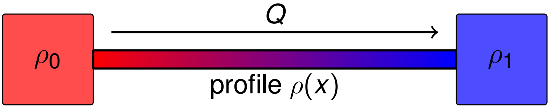

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$$\dots\langle\underline{z_1}|e^{i\Delta t\mathbb{H}}|\text{initial}\rangle$$

$$= \int \mathcal{D}p\mathcal{D}q \ \exp\{i\tfrac{1}{\hbar}\underbrace{\mathcal{S}[p,q]}_{\text{action}}\}$$

Macroscopic limit

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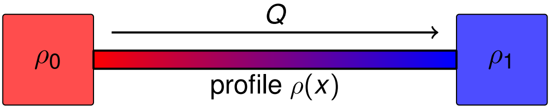

Using $SU(2)$ coherent states:

$$\langle \rho_{\mathrm{f}} | e^{t\mathbb{W}} | \rho_{\mathrm{i}} \rangle = \int_{\rho(0)=\rho_{\mathrm{i}}}^{\rho(t)=\rho_{\mathrm{f}}} \mathcal{D}\rho \mathcal{D}\hat{\rho} \; \exp\{ L \underbrace{\mathcal{S}[\hat{\rho},\rho]}_{\text{action}} \}$$

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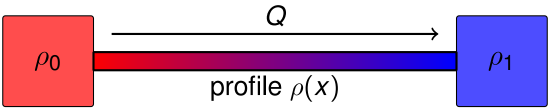

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$$\langle e^{-sQ} \rangle \sim \langle \rho_{\rm f} | e^{t\mathbb{W}_s} | \rho_{\rm i} \rangle = \int_{\rho(0)=\rho_{\rm i}}^{\rho(t)=\rho_{\rm f}} \mathcal{D}\rho \mathcal{D}\hat{\rho} \ \exp\{ L \underbrace{\mathcal{S}_s[\hat{\rho},\rho]}_{\text{action}} \}$$

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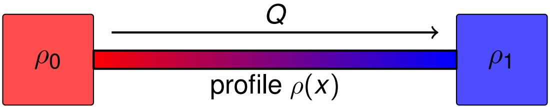

Same $\mathcal{S}_s[\hat{\rho}, \rho]$ as the MSR action of the Langevin evolution:

$$\partial_t \rho = -\partial_x \big[-\partial_x \rho + \xi \big]$$

$$\langle \xi(x,t)\xi(x',t') \rangle = \underbrace{\frac{1}{L}\rho(1-\rho)}_{\text{density-dependent}} \quad \delta(x'-x)\delta(t'-t)$$

Macroscopic limit

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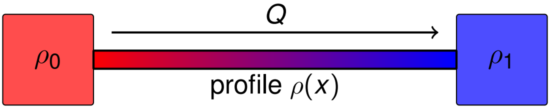

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One recovers the action of fluctuating hydrodynamics

[Spohn, Bertini De Sole Gabrielli Jona-Lasinio Landim]

Result

[Imparato, VL, van Wijland, **PRE** 80 011131]

- Large deviation function

[Reminder: $\langle e^{-sQ}\rangle \sim e^{t\psi(s)}$]

$$\psi(s) = \underbrace{\frac{1}{L}\mu(s)}_{\text{saddle}} + \underbrace{\frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(s)\right)}_{\text{fluctuations}}$$

$$\mu(s) = -\left(\operatorname{arcsinh}\sqrt{\omega}\right)^2 \qquad \omega = (1-e^{s})\left(e^{-s}\rho_0-\rho_1-(e^{-s}-1)\rho_0\rho_1\right)$$

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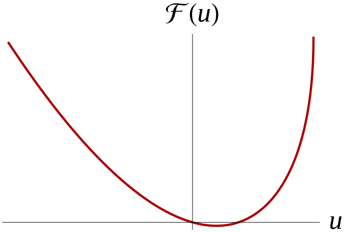
- Universal function $\mathcal{F}$

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$



Summary

Approach:

- operator formalism
- large deviation function

Extensions:

- fluctuating hydrodynamics
- saddle-point method, instantons
- integration of fluctuations (dynamical phase transition)

Open questions:

$\rightarrow$ Eq$\leftrightarrow$non-eq mapping in higher dimensions?
$\rightarrow$ More generic systems of interacting particle?
$\rightarrow$ Link to other eq$\leftrightarrow$non-eq mappings?
$\rightarrow$ Crossover to KPZ? Universal fluctuations?

Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

For **periodic boundary conditions**

- large deviation function

$$\psi(s) = \underbrace{L^{-1}\rho(1-\rho)s^2}_{\text{order 0}} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} \quad \text{with} \quad u = -\tfrac{1}{2}\rho(1-\rho)s^2$$

- universal function

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

- scaling of cumulants of the total current Q_{tot}

$$\tfrac{1}{t}\langle Q_{\text{tot}}^2\rangle \sim L$$

$$\tfrac{1}{t}\langle Q_{\text{tot}}^{2k}\rangle \sim L^{2k-2} \qquad (k \geq 2)$$

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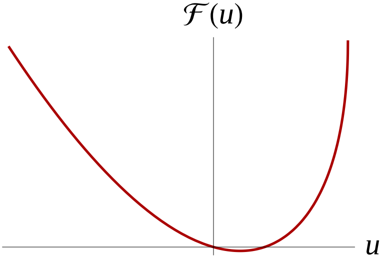


With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
Driving field E

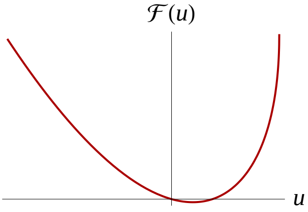
With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
Driving field E

For the WASEP:
$D = 1$, $\sigma = 2\rho(1-\rho)$

Large deviation function

$$\psi(s) = \underbrace{\tfrac{1}{2} s(s-E)\frac{\langle Q^2\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2} D\mathcal{F}(u)}_{\text{small fluctuations (determinant)}} \quad \text{with} \quad u = \underbrace{-s(s-E)\frac{\sigma\sigma''}{16D^2}}_{\text{can become } > 0}$$

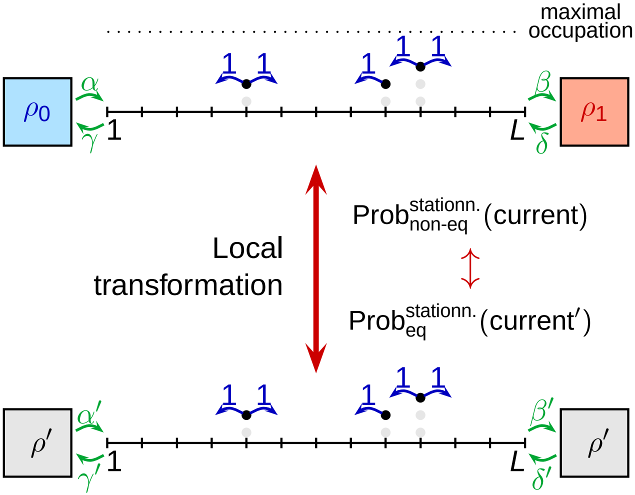


Dynamical phase transition
between
stationnary and non-stationnary
profiles

For the current

[Imparato, VL, van Wijland, **PRE** 80 011131]

Symmetric exclusion process



System in equilibrium

For the density profile

[Tailleur, Kurchan, VL, JPA **41** 505001]

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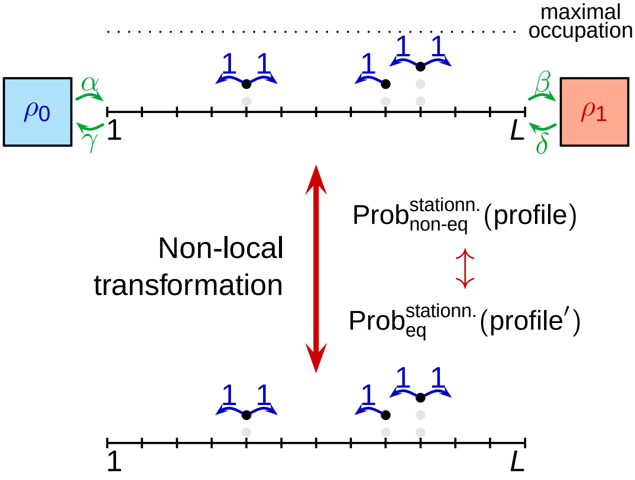
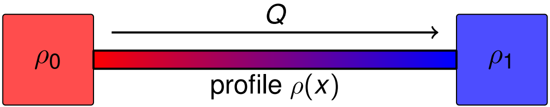


System in equilibrium

Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]

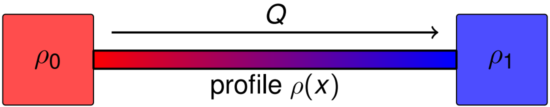


Boundary-driven transport model:

- long-range correlations
- breaking of time-reversal symmetry

Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

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Non-local mapping to equilibrium:

- accounts for long-range correlations
- (density gradient)$_{\text{non-eq.}} \longleftrightarrow$ (fixed density)$_{\text{eq.}}$
- yields Prob[$\rho(x)$] through a extremalization principle